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Autor: Laksov, D. / Svensson, L. / Thorup, A.
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THE SPECTRAL MAPPING THEOREM, NORMS ON RINGS, AND RESULTANTS

by D. LAKSOV, L. SVENSSON and A. THORUP

ABSTRACT. We give a short, simple and self-contained proof of the Spectral Mapping Theorem for matrices with entries in an arbitrary commutative ring. The result is placed in the wider framework of norms on algebras. It is shown that the Spectral Mapping Theorem follows from a uniqueness result for norms on polynomial rings in one variable. The results are used to generalize classical formulas for the resultant of polynomials.

1. INTRODUCTION

It is a well-known and useful result in spectral theory of complex finite dimensional vector spaces that if the characteristic polynomial of an $n \times n$-matrix M splits as $P_M(t) = \det(tI_n - M) = \prod_{i=1}^{n}(t - \lambda_i)$ then, for any polynomial $F(x)$, we have that $\det(tI_n - F(M)) = \prod_{i=1}^{n}(t - F(\lambda_i))$. We call this result the Spectral Mapping Theorem, because it is similar to the Spectral Mapping Theorem for Banach algebras. Many proofs of the result for complex finite dimensional vector spaces are known, most of them based upon transforming the matrix into triangular form (see [B2], §5, Proposition 10, p. 36), or using the Jordan canonical form for the matrix (see [L], Chapter XIV, §3, Theorem 3.10, p. 566). The Theorem and its proofs are easily generalized to arbitrary fields, and therefore to integral domains. In our work on parameter spaces in algebraic geometry ([L-S], [S1], [S2]) we needed a generalization of the Spectral Mapping Theorem to matrices with entries in arbitrary commutative rings with unity. The only reference we could find to such a generalization was [L], Chapter XIV, §3, Theorem 3.10, p. 566, where a proof is deduced from the theory of integral ring extensions. The difficult part of the proof is dismissed with the phrase "This is obvious to the reader who read the chapter on integral ring extensions, and the reader

who has not can forget about this part of the theorem". It is hard to decide whether these assertions about a *general reader* are correct. There is little evidence of the second claim. For the first there is more evidence. Indeed, it is true that, in order to reduce to the case when the ring is an integral domain, it is not hard to see that it suffices to prove the following assertion: Let M be the generic matrix over $\mathbf{Q}$, and let L be the splitting field, over $\mathbf{Q}(M)$, of the characteristic polynomial of M. Then L has degree $n!$ over $\mathbf{Q}(M)$. The latter assertion follows, using standard methods, from the theory of integral Galois extensions (see e.g. [L], Chapter VII, §2, Proposition 2.5, p. 342). Apparently the methods are foreign to the problem, and the results on integral ring extensions that are used are more difficult than the result that we want to prove. In this article we follow a more natural path, resulting in a simple, self contained, and short proof of the Spectral Mapping Theorem. As a consequence we get a better understanding of the result and we can place it into a more general framework.

The proof suggests that the Spectral Mapping Theorem should be considered within the framework of norms on algebras. Our method leads to a uniqueness result for norms on the polynomial ring in one variable from which a generalized Spectral Mapping Theorem follows. Applied to the most common norms on the polynomial ring in one variable the uniqueness gives generalizations of classical formulas for the resultant of polynomials.

2. The Spectral Mapping Theorem

Let M be an $n \times n$-matrix with entries in a commutative ring k with unity. Assume that the characteristic polynomial $P_M(t) = \det(tI_n - M)$ of M splits completely in k, that is $P_M(t) = \prod_{i=1}^{n}(t - \lambda_i)$ with $\lambda_i \in k$ for $i = 1, \ldots, n$.

The *Spectral Mapping Theorem* states that, for every polynomial $F(x)$ in the variable x with coefficients in an arbitrary commutative ring R that contains k as a subring, we have

$$\det F(M) = \prod_{i=1}^{n} F(\lambda_i) \tag{2.1}$$

in R. In particular, when $f(x)$ is a polynomial with coefficients in k and we use (2.1) for the ring $k[t]$ and the polynomial $F(x) = t - f(x)$, we obtain in $k[t]$:

$$P_{f(M)}(t) = \det(tI_n - f(M)) = \det F(M) = \prod_{i=1}^{n} F(\lambda_i) = \prod_{i=1}^{n}(t - f(\lambda_i)).$$