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Artikel: THE SPECTRAL MAPPING THEOREM, NORMS ON RINGS, AND RESULTANTS

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Kapitel: 3. A REDUCTION

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3. A REDUCTION

To prove the Spectral Mapping Theorem it suffices to verify that it holds for the polynomial ring $k[t_0,\dots,t_m]$ in variables $t_0,\dots,t_m$ over k, and for the polynomial $F(x)=t_0+t_1x+\cdots+t_mx^m$. This is because any polynomial $G(x)=b_0+b_1x+\cdots+b_mx^m$ with coefficients in a ring R containing k as a subring is the image of $F(x)$ by the map $g\colon k[t_0,\dots,t_m][x]\to R[x]$ defined by $g(a)=a$ for $a\in k$, by $g(t_i)=b_i$ for $i=0,\dots,m$, and by $g(x)=x$. If we can prove the equality $\det F(M)=\prod_{i=1}^n F(\lambda_i)$ in $k[t_0,\dots,t_m]$ we obtain that $\det G(M)=g(\det F(M))=\prod_{i=1}^n g(F(\lambda_i))=\prod_{i=1}^n G(\lambda_i)$ in R.

4. THE PROOF

Clearly (2.1) holds when F is a constant a where it simply states that $\det(aI_n)=a^n$. We shall prove (2.1) for polynomials F of degree $m>0$ by induction on m.

We first note that if $F(x)$ has a root λ in R, so that $F(x)=(x-\lambda)G(x)$ in $R[x]$, then (2.1) holds for $F(x)$. Indeed, $G(x)$ is of degree $m-1$ so it follows from the induction hypothesis that $\det G(M)=\prod_{i=1}^n G(\lambda_i)$. Since $F(M)=(M-\lambda I_n)G(M)$ we obtain:

$$\begin{aligned}\det F(M)&=\det(M-\lambda I_n)\det G(M)\\&=\prod_{i=1}^n(\lambda_i-\lambda)\prod_{i=1}^n G(\lambda_i)=\prod_{i=1}^n(\lambda_i-\lambda)G(\lambda_i)=\prod_{i=1}^n F(\lambda_i).\end{aligned}$$

As we saw in Section 3 it suffices to prove the result for the ring $Q=k[t_0,\dots,t_m]$ and the polynomial $F(x)=t_0+t_1x+\cdots+t_mx^m$. Let x and y be independent variables over the ring Q. The polynomial $F(x)-F(y)$ in x with coefficients in $Q[y]$ has the root $x=y$. Hence, as we just observed, (2.1) holds for the polynomial $F(x)-F(y)$. We obtain the equation:

$$\det(F(M)-F(y)I_n)=\prod_{i=1}^n(F(\lambda_i)-F(y)) \tag{4.1}$$

in $Q[y]$.

The equation (2.1) is a consequence of (4.1). To see this we observe that $F(y)$ in $Q[y]$ is transcendent over Q, that is the element $F(y)$ in $Q[y]$ does not satisfy a polynomial relation $a_0+a_1F(y)+\cdots+a_lF(y)^l=0$ with coefficients a_i in Q and $a_l\neq 0$, because the coefficient $a_lt_m^l$ of the highest