# 5. Norms on algebras

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power $y^{ml}$ of $y$ that appears in the relation is non-zero. It follows that we can define a homomorphism of rings $h\colon Q[F(y)] \to Q$ by $h(a) = a$ for $a \in Q$, and $h(F(y)) = 0$. We apply the map $h$ to both sides of (4.1) and obtain the equality (2.1).

## 5. Norms on algebras

The only properties of determinants that we used in the proof of the Spectral Mapping Theorem is that they are multiplicative, functorial and homogeneous. It is therefore natural to place the proof into the more general framework of norms on algebras. The advantage of this point of view is that we obtain a deeper understanding of the Spectral Mapping Theorem, and we obtain a natural connection with resultants of polynomials.

A *norm* $N$ of *degree* $n$ on a, not necessarily commutative, $k$-algebra $A$ is a family of maps $N_R\colon R \otimes_k A \to R$, one for every commutative $k$-algebra $R$, that satisfies the conditions:

(1) $N_R(a \otimes 1) = a^n$ for all elements $a$ in $R$.

(2) $N_R(uv) = N_R(u)N_R(v)$ for all elements $u$ and $v$ of $R \otimes_k A$.

(3) For every homomorphism $\varphi\colon R \to S$ of commutative $k$-algebras we have $\varphi N_R = N_S(\varphi \otimes \mathrm{id}_A)$.

A norm on an algebra may be described as a *multiplicative homogeneous polynomial law* (see Roby [R], or [B1], §9, Définition 3, p. 52).

For any map $B \to A$ of $k$-algebras the norm $N$ on $A$ *restricts* to a norm on $B$ of degree $n$. Moreover, for every homomorphism of commutative rings $k \to k'$ the norm $N$ on $A$ *induces* a norm of degree $n$ on the $k'$-algebra $k' \otimes_k A$.

Let $N$ be a norm of degree $n$ on a $k$-algebra $A$. Denote by $k[t]$ the $k$-algebra of polynomials in the variable $t$ with coefficients in $k$. For every element $\alpha$ in $A$ the polynomial in $k[t]$ :

$$P_\alpha(t) = P_\alpha^N(t) = N_{k[t]}(t - \alpha)$$

is called the *characteristic polynomial* of $\alpha$. The *trace* $\mathrm{Tr}^N(\alpha)$ of $\alpha$ is the element in $k$ such that $-\mathrm{Tr}^N(\alpha)$ is the coefficient of $t^{n-1}$ in $P_\alpha(t)$.

We note that $P_\alpha(0) = (-1)^n N_k(\alpha)$.

5.1. LEMMA. *Let $N$ be a norm of degree $n$ on a $k$-algebra $A$. For each element $\alpha$ of $A$ the characteristic polynomial $P_\alpha^N(t) = N_{k[t]}(t-\alpha)$ is monic of degree $n$.*

*Moreover, the trace* $\mathrm{Tr}^N$ *is a $k$-linear map $A \to k$.*

*Proof.* Let $s, t, u, v$ be independent variables over the ring $k$. For each element $\beta$ in $A$ the norm $N_{k[s,t,u]}(t - \alpha s - \beta u)$ is a polynomial in $k[s,t,u]$. Since $N$ is of degree $n$ we have that $N_{k[s,t,u,v]}(vt - \alpha vs - \beta vu) = v^n N_{k[,s,t,u]}(t - \alpha s - \beta u)$. It follows that $N_{k[s,t,u]}(t - \alpha s - \beta u)$ is homogeneous of degree $n$ in $k[s,t,u]$. In particular the coefficient of $t^{n-1}$ is of the form $as + bu$ with $a$ and $b$ in $k$. By evaluating the polynomial $N_{k[s,t,u]}(t - \alpha s - \beta u)$ at $s = 0$, $u = 0$, it follows that the coefficient to $t^n$ is equal to $1$. Hence $N_{k[t]}(t-\alpha)$ is a monic polynomial of degree $n$, and $a = -\mathrm{Tr}^N(\alpha)$. Similarly, $b = -\mathrm{Tr}^N(\beta)$. Hence we have that $\mathrm{Tr}^N(\alpha s + \beta u) = -(as + bu) = \mathrm{Tr}^N(\alpha)s + \mathrm{Tr}^N(\beta)t$. Specializing $s$ and $t$ to any pair of elements of $k$ the second part of the Lemma follows. □

5.2. EXAMPLE. Let $M$ be a free module of rank $n$ over $k$, or more generally a projective $k$-module of constant rank $n$. Then the determinant defines a norm of degree $n$ on $\mathrm{End}_k(M)$.

Let $A$ be a $k$-algebra which is free of rank $n$ as a $k$-module. Left multiplication by elements of $A$ define an injection $A \to \mathrm{End}_k(A)$ of $k$-algebras. By restriction we obtain a norm of degree $n$ on $A$.

## 6. NORMS AND RESULTANTS

Let $F(x) = f_0 + \cdots + f_m x^m$ and $P(x) = p_0 + \cdots + p_n x^n$ be polynomials of degree $m$, respectively $n$ in the $k$-algebra $k[x]$ of polynomials in the variable $x$ with coefficients in $k$. The *resultant* $\mathrm{Res}(F,P)$ of $F$ and $P$ is the determinant of the $(m+n) \times (m+n)$-matrix $D(F,P)$ whose columns are the coefficients of the polynomials $F, xF, \ldots, x^{n-1}F, P, xP, \ldots, x^{m-1}P$. Note that the definition is asymmetric in $F$ and $P$ in the sense that $\mathrm{Res}(F,P) = (-1)^{mn}\mathrm{Res}(P,F)$.

When $P$ is monic the resultant is equal to the determinant of the endomorphism induced by multiplication by $F$ on the free $k$-module $k[x]/(P(x))$ of rank $n$. To see this we note that for $i = 0, \ldots, n-1$ we can write $x^i F = Q_i P + R_i$ in $k[x]$, where $Q_i(x)$ and $R_i(x)$ are of degrees at most $m-1$, respectively $n-1$. It follows that the determinant of $D(F,P)$ is equal to the determinant of the $(m+n) \times (m+n)$-matrix $B(F,P)$ whose columns are the