

6. NORMS AND RESULTANTS

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5.1. LEMMA. Let N be a norm of degree n on a k -algebra A . For each element α of A the characteristic polynomial $P_\alpha^N(t) = N_{k[t]}(t - \alpha)$ is monic of degree n .

Moreover, the trace Tr^N is a k -linear map $A \rightarrow k$.

Proof. Let s, t, u, v be independent variables over the ring k . For each element β in A the norm $N_{k[s,t,u]}(t - \alpha s - \beta u)$ is a polynomial in $k[s, t, u]$. Since N is of degree n we have that $N_{k[s,t,u,v]}(vt - \alpha vs - \beta vu) = v^n N_{k[s,t,u]}(t - \alpha s - \beta u)$. It follows that $N_{k[s,t,u]}(t - \alpha s - \beta u)$ is homogeneous of degree n in $k[s, t, u]$. In particular the coefficient of t^{n-1} is of the form $as + bu$ with a and b in k . By evaluating the polynomial $N_{k[s,t,u]}(t - \alpha s - \beta u)$ at $s = 0, u = 0$, it follows that the coefficient to t^n is equal to 1. Hence $N_{k[t]}(t - \alpha)$ is a monic polynomial of degree n , and $a = -\text{Tr}^N(\alpha)$. Similarly, $b = -\text{Tr}^N(\beta)$. Hence we have that $\text{Tr}^N(\alpha s + \beta u) = -(as + bu) = \text{Tr}^N(\alpha)s + \text{Tr}^N(\beta)t$. Specializing s and t to any pair of elements of k the second part of the Lemma follows. \square

5.2. EXAMPLE. Let M be a free module of rank n over k , or more generally a projective k -module of constant rank n . Then the determinant defines a norm of degree n on $\text{End}_k(M)$.

Let A be a k -algebra which is free of rank n as a k -module. Left multiplication by elements of A define an injection $A \rightarrow \text{End}_k(A)$ of k -algebras. By restriction we obtain a norm of degree n on A .

6. NORMS AND RESULTANTS

Let $F(x) = f_0 + \cdots + f_m x^m$ and $P(x) = p_0 + \cdots + p_n x^n$ be polynomials of degree m , respectively n in the k -algebra $k[x]$ of polynomials in the variable x with coefficients in k . The *resultant* $\text{Res}(F, P)$ of F and P is the determinant of the $(m+n) \times (m+n)$ -matrix $D(F, P)$ whose columns are the coefficients of the polynomials $F, xF, \dots, x^{n-1}F, P, xP, \dots, x^{m-1}P$. Note that the definition is asymmetric in F and P in the sense that $\text{Res}(F, P) = (-1)^{mn} \text{Res}(P, F)$.

When P is monic the resultant is equal to the determinant of the endomorphism induced by multiplication by F on the free k -module $k[x]/(P(x))$ of rank n . To see this we note that for $i = 0, \dots, n-1$ we can write $x^i F = Q_i P + R_i$ in $k[x]$, where $Q_i(x)$ and $R_i(x)$ are of degrees at most $m-1$, respectively $n-1$. It follows that the determinant of $D(F, P)$ is equal to the determinant of the $(m+n) \times (m+n)$ -matrix $B(F, P)$ whose columns are the

coefficients of the polynomials $R_0, \dots, R_{n-1}, P, xP, \dots, x^{m-1}P$. We see that the $n \times n$ -block $C(F, P)$ in the upper left corner of $B(F, P)$ is the matrix $C(F, P)$ of the map induced by multiplication by F on $k[x]/(P(x))$, and the $m \times m$ -block in the lower right corner is upper triangular with 1's on the diagonal. Moreover, the entries of $C(F, P)$ are the only non-zero entries in the first n columns of $B(F, P)$. It follows that $\text{Res}(F, P) = \det C(F, P)$, as we claimed.

6.1. EXAMPLE. When P is a monic polynomial we saw in Example 5.2 that the k -algebra $k[x]/(P(x))$ which is free of rank n as a k -module has a canonical norm. Via the canonical map $k[x] \rightarrow k[x]/(P(x))$ we obtain a canonical norm N'_P on $k[x]$. The above interpretation of the resultant can then be written as

$$(6.1.1) \quad (N'_P)_R(F) = \text{Res}(F, P)$$

for all commutative k -algebras R and all polynomials $F(x)$ in $R[x] = R \otimes_k k[x]$. By an easy computation of the determinant defining $\text{Res}(t - x, P)$, we obtain that the characteristic polynomial of x with respect to N'_P is

$$P_x^{N'_P}(t) = P(t).$$

6.2. EXAMPLE. We shall introduce a second important norm on $k[x]$. Let $P(x)$ be a monic polynomial of degree n in the k -algebra $k[x]$. There is a canonical ring extension $k \subseteq k' = k[\lambda_1, \dots, \lambda_n]$ such that $P(x)$ splits as $P(x) = \prod_{i=1}^n (x - \lambda_i)$ in $k'[x]$. The extension is obtained by induction starting with $k = k_0$ and $P_0(x) = P(x)$, and constructing $k_i = k[\lambda_1, \dots, \lambda_i]$ and $P_i(x) \in k[\lambda_1, \dots, \lambda_i][x]$ from k_{i-1} and P_{i-1} , for $i = 1, 2, \dots, n$, by $k_i = k_{i-1}[x]/(P_{i-1}(x)) = k_{i-1}[\lambda_i]$, where λ_i is the class of x , and by $P_i(x) = P_{i-1}(x)/(x - \lambda_i)$. We note that k' is a free k -module of rank $n!$. The algebra k' is sometimes called the *universal decomposition algebra* for P (see [B1], §6, p. 68).

For every commutative k -algebra R and every polynomial G in $R[x] = R \otimes_k k[x]$ we have that $\prod_{i=1}^n G(\lambda_i)$ is symmetric in $\lambda_1, \dots, \lambda_n$, and consequently lies in the image of the inclusion $R \subseteq k' \otimes_k R$. We obtain a map $(N''_P)_R: R \otimes_k k[x] \rightarrow R$ defined by $(N''_P)_R(G) = \prod_{i=1}^n G(\lambda_i)$. In this way we obtain a norm N''_P of degree n on $k[x]$ and the characteristic polynomial of x with respect to the norm N''_P is

$$P_x^{N''_P}(t) = \prod_{i=1}^n (t - \lambda_i) = P(t).$$