7. Uniqueness of norms and the Spectral Mapping Theorem

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7. UNIQUENESS OF NORMS AND THE SPECTRAL MAPPING THEOREM

We are now ready to prove the uniqueness of norms on the polynomial ring $k[x]$ that we alluded to in the introduction. When we apply uniqueness to the norms N'_P and N''_P of Examples 6.1 and 6.2 we obtain the generalization of the Spectral Mapping Theorem for norms of rings also mentioned in the introduction. We also obtain some generalizations to rings of the classical interpretations of the resultant.

The proof of the uniqueness result is a slight variation of the proof of the Spectral Mapping Theorem given in Section 4.

The first formula in (7.2.1) below, when the norm is the determinant on the algebra of $n \times n$ matrices over an arbitrary ring k, was proved by McCoy [M], Theorem 56, p. 172.

7.1. THEOREM. *A norm N on the k-algebra $k[x]$ of polynomials in the variable x is uniquely determined by the characteristic polynomial $P_x^N(t) = N_{k[t]}(t-x)$ of x with respect to N.*

Proof. Let N' be a second norm on $k[x]$ such that $P_x^N(t) = P_x^{N'}(t)$. Then N' and N are of the same degree n. The Theorem asserts that for any commutative k-algebra R and every polynomial F in $R[x] = R \otimes_k k[x]$ we have

$$N_R(F(x)) = N'_R(F(x)) \tag{7.1.1}$$

We prove (7.1.1) by induction on the degree m of $F(x)$. Clearly (7.1.1) holds when F is a constant. Assume that the degree m of F is positive and that the equality $N_R(G(x)) = N'_R(G(x))$ holds for all commutative k-algebras R and all polynomials $G(x)$ in $R[x]$ of degree $m-1$. Let t be an independent variable over $R[x]$. If the equality (7.1.1) holds for the polynomial $tx^m + F(x)$ in $R[t][x]$ it holds for $F(x)$, as we see by specializing t to 0. Consequently we may assume that the coefficient of x^m in $F(x)$ is a non-zero divisor in $R[x]$. Then the canonical map $R \to R[x]/(F(x))$ is an injection. Consequently we may also assume that R contains a root μ of $F(x)$. Then we have that $F(x) = F(x) - F(\mu) = (x-\mu)G(x)$ in $R[x]$ where $G(x)$ has degree $m-1$. Both sides of the equality (7.1.1) are multiplicative in $F(x)$, and the equality holds for $G(x)$ by the induction assumption. It also holds for $x-\mu$ as we see by specializing t to μ in the equality $N_{k[t]}(t-x) = P_x^N(t) = P_x^{N'}(t) = N'_{k[t]}(t-x)$. Hence we have proved the Theorem. □

7.2. COROLLARY (The Generalized Spectral Mapping Theorem). *Let N be a norm of degree n on a k-algebra A, and let α be an element of A. For all polynomials F in $k[x]$ we have the equations*

$$N_k(F(\alpha)) = \operatorname{Res}(F, P_\alpha^N) = \prod_{i=1}^{n} F(\lambda_i) \tag{7.2.1}$$

in k, where $\lambda_1, \ldots, \lambda_n$ are elements of any extension $R \supseteq k$ such that $P_\alpha^N(t) = \prod_{i=1}^n (t - \lambda_i)$ in $R[x]$.

In particular we have that $\operatorname{Tr}^N(F(\alpha)) = \sum_{i=1}^n F(\lambda_i)$.

Proof. Let $P = P_\alpha^N$ be the characteristic polynomial of α with respect to N. The norm N restricts, via the canonical k-algebra homomorphism $k[x] \to A$ which sends x to α, to a norm on $k[x]$, and the characteristic polynomial of x with respect to this norm is P. On $k[x]$ we have the norm N, and the norms N_P' and N_P'' of the Examples 6.1 and 6.2, and the characteristic polynomial of x with respect to all three norms is P. It follows from the Theorem that these three norms are equal. The equations (7.2.1) express the equality of the norms applied to the polynomial $F(x)$. Finally the expression for the trace follows by considering the coefficient of t^{n-1} of the left and right side of (7.2.1) applied to the polynomial $t - F(x)$ in $k[t][x]$. $\square$

The formula $\operatorname{Res}(F, P) = \prod_{i=1}^n F(\lambda_i)$ of Corollary (7.2) is the generalization to rings of the well-known interpretation of resultants by the roots of the monic polynomial P in the case when k is a field. If F is also monic and $F = \prod_{j=1}^m (x - \mu_j)$ in $R[x]$ we have

$$\operatorname{Res}(F, P) = \prod_{i=1}^{n} F(\lambda_i) = \prod_{i=1}^{n}\prod_{j=1}^{m}(\lambda_i - \mu_j)$$

which is often used as a definition of the resultant in the case k is an algebraically closed field.

8. THE DISCRIMINANT

We shall use the Generalized Spectral Mapping Theorem of Section 7 to prove two results on discriminants that are well-known for algebras of finite dimension over fields (see e.g. [B2], §5, Corollaire 6 and Corollaire 7, p. 38). Note that k below, as above, denotes a commutative ring with unity.