8. The discriminant

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

7.2. COROLLARY (The Generalized Spectral Mapping Theorem). *Let N be a norm of degree n on a k-algebra A, and let α be an element of A. For all polynomials F in $k[x]$ we have the equations*

$$N_k(F(\alpha)) = \operatorname{Res}(F, P_\alpha^N) = \prod_{i=1}^{n} F(\lambda_i) \tag{7.2.1}$$

in k, where $\lambda_1, \ldots, \lambda_n$ are elements of any extension $R \supseteq k$ such that $P_\alpha^N(t) = \prod_{i=1}^n (t - \lambda_i)$ in $R[x]$.

In particular we have that $\operatorname{Tr}^N(F(\alpha)) = \sum_{i=1}^n F(\lambda_i)$.

Proof. Let $P = P_\alpha^N$ be the characteristic polynomial of α with respect to N. The norm N restricts, via the canonical k-algebra homomorphism $k[x] \to A$ which sends x to α, to a norm on $k[x]$, and the characteristic polynomial of x with respect to this norm is P. On $k[x]$ we have the norm N, and the norms N'_P and N''_P of the Examples 6.1 and 6.2, and the characteristic polynomial of x with respect to all three norms is P. It follows from the Theorem that these three norms are equal. The equations (7.2.1) express the equality of the norms applied to the polynomial $F(x)$. Finally the expression for the trace follows by considering the coefficient of t^{n-1} of the left and right side of (7.2.1) applied to the polynomial $t - F(x)$ in $k[t][x]$. □

The formula $\operatorname{Res}(F, P) = \prod_{i=1}^n F(\lambda_i)$ of Corollary (7.2) is the generalization to rings of the well-known interpretation of resultants by the roots of the monic polynomial P in the case when k is a field. If F is also monic and $F = \prod_{j=1}^m (x - \mu_j)$ in $R[x]$ we have

$$\operatorname{Res}(F, P) = \prod_{i=1}^{n} F(\lambda_i) = \prod_{i=1}^{n} \prod_{j=1}^{m} (\lambda_i - \mu_j)$$

which is often used as a definition of the resultant in the case k is an algebraically closed field.

8. THE DISCRIMINANT

We shall use the Generalized Spectral Mapping Theorem of Section 7 to prove two results on discriminants that are well-known for algebras of finite dimension over fields (see e.g. [B2], §5, Corollaire 6 and Corollaire 7, p. 38). Note that k below, as above, denotes a commutative ring with unity.

Let P be a monic polynomial in $k[x]$ of degree n. The *discriminant* of P is the element $(-1)^{n(n-1)/2}\operatorname{Res}(P', P)$ of k, where P' is the derivative of P.

Let $k \subseteq k' = k[\lambda_1, \dots, \lambda_n]$ be the canonical extension constructed in Example 6.2. We write $\Delta(P) = \prod_{i>j}(\lambda_i - \lambda_j)$. The formula for the resultant of Corollary 7.2 gives the equations

$$(-1)^{n(n-1)/2}\operatorname{Res}(P', P) = (-1)^{n(n-1)/2}\prod_{i\neq j}(\lambda_i - \lambda_j) = \prod_{i>j}(\lambda_i - \lambda_j)^2 = \Delta(P)^2,$$

hence $\Delta(P)^2$ is the discriminant of P.

8.1. PROPOSITION. *Let N be a norm of degree n on a k-algebra A, and let α be an element of A. Denote by P_α the characteristic polynomial of α with respect to N. Then we have the equations in k:*

$$\det \operatorname{Tr}^N(\alpha^{p+q})_{p,q=0,\dots,n-1} = \Delta(P_\alpha)^2 = (-1)^{n(n-1)/2}N_k(P'_\alpha(\alpha)).$$

Proof. By (7.2.1) we have the equation $N_k(P_\alpha(\alpha)) = \operatorname{Res}(P'_\alpha, P_\alpha)$. Hence the second equation holds.

The rest of the proof is classical, as given in [B2]. We note that $\prod_{i>j}(\lambda_i - \lambda_j)$ is the determinant of the matrix (λ_i^q) with row number $i = 1, \dots, n$ and column number $q = 0, \dots, n-1$. When this matrix is multiplied from the left by its transpose the entry in position p, q is equal to the sum $\sum_{i=1}^n \lambda_i^{p+q}$, and consequently equal to $\operatorname{Tr}^N(\alpha^{p+q})$. Hence we have proved the Proposition. □

8.2. PROPOSITION. *Let N be a norm on a k-algebra A, and let α be an element in A. In the ring of power series in the variable t with coefficients in k we have the equation:*

$$(\tfrac{d}{dt}\log)N_{k[t]}(1 - t\alpha) := \tfrac{d}{dt}N_{k[t]}(1 - t\alpha)/N_{k[t]}(1 - t\alpha) = -\sum_{j=0}^{\infty}\operatorname{Tr}^N(\alpha^{j+1})t^j.$$

Proof. Let n be the degree of the norm N. We have

$$N_{k[t]}(1 - t\alpha) = \prod_{i=1}^{n}(1 - \lambda_i t)$$

and consequently we have equations

$$(d\log/dt)N_{k[t]}(1 - t\alpha) = \sum_{i=1}^{n} -\lambda_i/(1 - t\lambda_i) = -\sum_{j=0}^{\infty}\sum_{i=1}^{n}\lambda_i^{j+1}t^j.$$

It follows from Corollary 7.2 that $\operatorname{Tr}^N(\alpha^l) = \sum_{i=1}^n \lambda_i^l$ for $l = 1, 2, \dots$. The formula of the Proposition follows. □

We note that, when k contains the rational numbers and N is a norm of degree n on a k-algebra A, we have that the characteristic polynomial $N_{k[t]}(t-\alpha)$ of an element α of A is determined by the elements $\mathrm{Tr}^N(\alpha^j)$ for $j=1,\ldots,n$.

REFERENCES

[B1] BOURBAKI, N. *Algèbre.* Chapitre IV, Polynômes et fractions rationnelles. Masson, Paris, 1981.

[B2] —— *Algèbre.* Chapitre VII, Modules sur les anneaux principaux. Masson, Paris, 1981.

[L-S] LAKSOV, D. and R. M. SKJELNES. The Hilbert scheme parametrizing finite length subschemes of the line with support at the origin. To appear in *Compositio Math.*

[L] LANG, S. *Algebra.* 3rd edition. Addison-Wesley Publishing Company, 1993.

[M] MCCOY, N. H. *Rings and Ideals.* The Carus Mathematical Monographs 8. The Mathematical Association of America, 1948.

[R] ROBY, N. Lois polynômes et lois formelles en théorie des modules. *Ann. Sci. École Norm. Sup. 80* (1963), 213–248.

[S1] SKJELNES, R. M. On the representability of $Hilb^n k[x]_{(x)}$. To appear in *J. London Math. Soc.* (1999).

[S2] —— Symmetric tensors with applications to Hilbert schemes. To appear (1999).

(Reçu le 27 janvier 2000; version révisée reçue le 3 mai 2000)

Dan Laksov
L. Svensson

Department of Mathematics, KTH
KTH, S-100 44 Stockholm
Sweden
e-mail: laksov@math.kth.se
larss@math.kth.se

A. Thorup

Department of Mathematics, University of Copenhagen
Universitetsparken 5, DK-2100 København Ø,
Denmark
e-mail: thorup@math.ku.dk