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THE WITT GROUP OF LAURENT POLYNOMIALS

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ABSTRACT. We give a direct, self-contained proof of the fact that for a large class of rings A, in particular for all regular rings with involution, $W(A[t,1/t]) = W(A) \oplus W(A)$.

1. INTRODUCTION

The purpose of this note is to give a short direct proof of two fundamental theorems on the Witt group of polynomials and Laurent extensions of a ring A. These theorems were proved independently by M. Karoubi [3] and by A. Ranicki [5]. We will state them under the most general conditions on A and for their proofs we will use nothing more than a general result on the K-theory of Laurent polynomials. In the last section we will show, by two counterexamples, that the assumptions we make on A are necessary.

We begin by recalling briefly some definitions. We refer to [4] for a more detailed exposition and for the proofs of the few basic results that we will use.

Let A be an associative ring with an involution denoted by $a \mapsto a^\circ$. Except in §2 we will always assume that 2 is invertible in A. If M is a right A-module, we denote by M^* its dual $\mathrm{Hom}_A(M,A)$ endowed with the right action of A given by $fa(x) = a^\circ f(x)$ for any $f\colon M \to A$ and $a \in A$. If P is a finitely generated projective right A-module we identify it with P^{**} through the canonical isomorphism mapping $x \in P$ to $\widehat{x}\colon P^* \to A$ defined by $\widehat{x}(f) = f(x)$.

Let ϵ be 1 or -1. An *ϵ-hermitian space* over A is a pair (P,α) consisting of a finitely generated projective right A-module P and an A-isomorphism $\alpha\colon P \to P^*$ satisfying $\alpha = \epsilon\alpha^*$. For brevity ϵ-hermitian spaces will be called *spaces*. A 1-hermitian space (over a commutative ring A) is also called a *quadratic space*.

Two spaces (P, α) and (Q, β) are *isometric* if there exists an A-isomorphism $\varphi\colon P \to Q$ such that the square

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & Q \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P^* & \xleftarrow[\varphi^*]{} & Q^* \end{array}$$

commutes. A space is *hyperbolic* if it is isometric to a space of the form

$$H(P) = \left(P \oplus P^*, \left(\begin{smallmatrix} 0 & 1 \\ \epsilon & 0 \end{smallmatrix}\right)\right).$$

The *orthogonal sum* of two spaces (P, α) and (Q, β) is the space

$$(P, \alpha) \perp (Q, \beta) = (P \oplus Q, \alpha \oplus \beta).$$

If (P, α) is a space and M a submodule of P we denote by $M^\perp$ the orthogonal of M, defined by the exact sequence

$$0 \longrightarrow M^\perp \longrightarrow P \xrightarrow{i^* \circ \alpha} M^*,$$

where i^* is the dual of the inclusion $i\colon M \to P$. A submodule M of P is *totally isotropic* if $M \subseteq M^\perp$. A *sublagrangian* of a space (P, α) is a totally isotropic direct factor of P. A *lagrangian* of (P, α) is a sublagrangian L such that $L = L^\perp$. For instance, P and P^* are lagrangians of $H(P)$.

The Witt group $W(A)$ of ϵ-hermitian spaces over A is the quotient of the Grothendieck group of ϵ-hermitian spaces with respect to orthogonal sums, by the subgroup generated by all hyperbolic spaces. We say that two spaces are *Witt equivalent* if they represent the same element of $W(A)$.

Consider now the rings $A[t]$ and $A[t, t^{-1}]$, endowed with the involution that fixes t and maps $a \in A$ to a°. For the ring $A[t, t^{-1}]$ we introduce a variant $W'(A[t, t^{-1}])$ of the Witt group. We first consider the Grothendieck group Q of ϵ-hermitian spaces over $A[t, t^{-1}]$ which are extended from A as $A[t, t^{-1}]$-modules, and its subgroup N generated by the hyperbolic spaces $H(P)$ where P is extended from A. We then define $W'(A[t, t^{-1}])$ as Q/N. Clearly $W'(A[t, t^{-1}])$ maps canonically to $W(A[t, t^{-1}])$. Here are our results.

A (THEOREM 3.1). *Let A be an associative ring with involution, in which 2 is invertible. The canonical homomorphism*

$$W(A) \to W(A[t])$$

is an isomorphism.

B (THEOREM 5.1). *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi\colon W(A)\oplus W(A)\to W'(A[t,t^{-1}])$$

mapping (ξ,η) to $\xi+t\eta$ is an isomorphism.

C (THEOREM 7.1). *Let A be an associative ring with involution, in which 2 is invertible. Let*

$$\varphi\colon W'(A[t,t^{-1}])\to W(A[t,t^{-1}])$$

be the canonical homomorphism.

(a) *If $H^2(\mathbf{Z}/2, K_{-1}(A))=0$, then φ is surjective.*

(b) *If $K_0(A)=K_0(A[t])=K_0(A[t,t^{-1}])$, then φ is an isomorphism.*

Two examples will be constructed in § 8 to show that the assumptions in (a) and in (b) cannot be omitted.

An amusing application of **B** is the following result:

D (PROPOSITION 6.8). *Let A be a commutative semilocal ring in which 2 is invertible. Let (P,α) be a quadratic space over A. If (P,α) is isometric to $(P,t\cdot\alpha)$ over $A[t,t^{-1}]$, then (P,α) is hyperbolic.*

We remark that in general, even for a commutative local ring, there is no residue map

$$Res\colon W(A[t,t^{-1}])\to W(A)$$

satisfying the following two properties:

- For any constant space $\xi\in W(A)\subset W(A[t,t^{-1}])$, $Res(\xi)=0$.
- For any constant space $\xi\in W(A)\subset W(A[t,t^{-1}])$, $Res(t\cdot\xi)=\xi$.

In fact, the existence of such a residue map immediately implies the injectivity of

$$\varphi\circ\psi\colon W(A)\oplus W(A)\to W(A[t,t^{-1}]),$$

which may fail, as in Example 8.1. However, there exists a residue map $Res\colon W'(A[t,t^{-1}])\to W(A)$ (Proposition 5.2) which yields the injectivity of ψ.

We now recall three elementary, well-known facts about hermitian spaces.

PROPOSITION 1.5. *Let (P,α) be any space. Then:*

1. *The space $(P,\alpha) \perp (P,-\alpha)$ is hyperbolic.*
2. *If L is a lagrangian of (P,α), then (P,α) is isometric to $H(L)$.*
3. *If M is a sublagrangian of (P,α), then the map α induces on $M^{\perp}/M$ a natural structure of hermitian space that makes it Witt equivalent to (P,α).*

2. K-THEORETIC PRELIMINARIES

We recall a few results proved in the twelfth chapter of Bass' book [1]. For any ring A we denote by $K_0(A)$ the Grothendieck group of finitely generated projective right A-modules and by $K_1(A)$ the abelianized general linear group of A: $K_1(A) = GL(A)/[GL(A), GL(A)]$. By Whitehead's lemma $K_1(A)$ is also the quotient of $GL(A)$ by the subgroup $E(A)$ generated by all elementary matrices over A.

For any functor F from rings to abelian groups we denote by $N_+F(A)$ the kernel of the map $F(A[t]) \to F(A)$ obtained by putting $t=0$. Similarly, we denote by $N_-F(A)$ the kernel of $F(A[t^{-1}]) \to F(A)$ obtained by putting $t^{-1}=0$. The inclusions of $A[t]$ and $A[t^{-1}]$ into $A[t,t^{-1}]$ define a map

$$N_+F(A) \oplus N_-F(A) \longrightarrow F(A[t,t^{-1}])$$

whose cokernel will be denoted by $LF(A)$. The functor LK_1 turns out to be naturally isomorphic to K_0, hence we will denote LK_i by K_{i-1} for $i=1$ and also for $i=0$.

THEOREM 2.1. *Let A be any associative ring.*
(a) *For $i=0$ or 1 there exists a natural embedding*

$$\lambda_i \colon K_{i-1}(A) \longrightarrow K_i(A[t,t^{-1}])$$

such that the composite

$$K_{i-1}(A) \xrightarrow{\lambda_i} K_i(A[t,t^{-1}]) \to LK_i(A) = K_{i-1}(A)$$

is the identity.