

4. The Witt group of torsion modules

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Proof of the lemma. Write $\alpha = \gamma + \delta t^N$, where δ is constant and γ of degree less than N . Assume that N is at least 2. Since δ is ϵ -hermitian and 2 is invertible in A we can write $\delta = \sigma + \epsilon\sigma^*$. Then

$$\begin{pmatrix} 1 & t & -\sigma^*t^{N-1} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma + \sigma t^N + \epsilon\sigma^*t^N & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \epsilon \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ t & 1 & 0 \\ -\sigma t^{N-1} & 0 & 1 \end{pmatrix}$$

is of degree $\leq N - 1$ and after $N - 1$ such transformations we get a linear matrix. \square

Writing $\alpha = \alpha_0 + t\alpha_1$ as $\alpha_0(1 + \nu t)$ we see immediately that, α being invertible, ν is nilpotent. The formal power series

$$\tau = (1 + \nu t)^{-1/2} = \sum \binom{-1/2}{k} (\nu t)^k$$

is a polynomial. From $\alpha = \epsilon\alpha^*$ we get $\alpha_0^* = \epsilon\alpha_0$ and $\nu^*\alpha_0^* = \epsilon\alpha_0\nu$. This implies that $\tau^*\alpha_0^* = \epsilon\alpha_0\tau$ and therefore

$$\tau^*\alpha\tau = \tau^*\alpha_0(1 + \nu t)\tau = \alpha_0\tau(1 + \nu t)\tau = \alpha_0.$$

This proves that (P, α) is Witt equivalent to $(P(0), \alpha(0))$ and is, therefore, hyperbolic. \square

4. THE WITT GROUP OF TORSION MODULES

Let M be a finitely generated right $A[t]$ -module and suppose that it is a t -torsion module and that it is projective as an A -module. Obviously, it will be finitely generated over A . We denote by M^\sharp the left $A[t]$ -module $\text{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t])$ and we consider it as a right module through the involution on $A[t]$.

Recall that, as an A -module, the quotient $A[t, t^{-1}]/A[t]$ can be written as a direct sum

$$A[t, t^{-1}]/A[t] = At^{-1} \oplus At^{-2} \oplus \dots$$

Thus, to any $f \in \text{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t])$ we can associate an A -linear map $f_{-1}: M \rightarrow A$, which is defined as the composite of f with the projection onto At^{-1} .

PROPOSITION 4.1. *The map*

$$\partial = \partial_M: M^\sharp = \text{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \longrightarrow \text{Hom}_A(M, A) = M^*$$

obtained by associating f_{-1} to f is a functorial A -linear isomorphism.

Proof. It is clear that ∂ is A -linear. To show that it is bijective we construct its inverse. Given any $g \in M^*$ define \tilde{g} by the (finite!) sum

$$\tilde{g}(x) = t^{-1}g(x) + t^{-2}g(tx) + t^{-3}g(t^2x) + \dots$$

It is easy to check that $\tilde{g} \in M^\sharp$, $(\tilde{g})_{-1} = g$ and $\tilde{f}_{-1} = f$. Functoriality is clear. \square

COROLLARY 4.2. *For any finitely generated t -torsion module M which is projective as an A -module the canonical homomorphism $M \rightarrow M^{\sharp\sharp}$ is an isomorphism.*

Proof. It suffices to remark that the diagram

$$\begin{array}{ccc} & M & \\ \text{can} \swarrow & & \searrow \text{can} \\ M^{\sharp\sharp} & \xrightarrow{(\partial_M^*)^{-1} \circ \partial_{M^\sharp}} & M^{**} \end{array}$$

commutes and that $M \xrightarrow{\text{can}} M^{**}$ is an isomorphism. \square

An ϵ -hermitian t -torsion space (or, briefly, a t -torsion space) is a pair (M, \langle, \rangle) consisting of a finitely generated t -torsion right $A[t]$ -module M which is projective as an A -module, and a perfect ϵ -hermitian pairing $\langle, \rangle: M \times M \rightarrow A[t, t^{-1}]/A[t]$. Giving \langle, \rangle is the same, of course, as giving its adjoint $\varphi: M \rightarrow M^\sharp$ defined by $\varphi(a)(b) = \langle a, b \rangle$.

Isometries and orthogonal sums are defined in the obvious way. For any subset $X \subset M$ we define its orthogonal as

$$X^\perp = \{y \in M \mid \langle x, y \rangle = 0 \quad \forall x \in X\}.$$

A *sublagrangian* of (M, φ) is an $A[t]$ -submodule L of M which satisfies the following two conditions:

- (1) It is contained in its own orthogonal: $L \subseteq L^\perp$.
- (2) The quotient M/L is projective over A (which is the same as saying that L , as an A -module, is a direct factor of M).

A sublagrangian L is a *lagrangian* if $L = L^\perp$. A t -torsion space is *metabolic* if it has a lagrangian. The Witt group of t -torsion spaces is the quotient of the Grothendieck group of t -torsion spaces with respect to orthogonal sums, modulo the subgroup generated by the metabolic spaces. We will denote it by $W_{tors}(A[t])$. Lemma 4.6 below will show that the opposite of the class of (M, φ) is the class of $(M, -\varphi)$.

LEMMA 4.3. *Let M and N be finitely generated t -torsion modules and $i: N \rightarrow M$ an $A[t]$ -linear homomorphism. Assume that as A -modules M and N are projective. Then the map $i^\sharp: M^\sharp \rightarrow N^\sharp$ is surjective (respectively injective) if and only if $i^*: M^* \rightarrow N^*$ is surjective (respectively injective).*

Proof. Look:

$$\begin{array}{ccc} M^\sharp & \xrightarrow{i^\sharp} & N^\sharp \\ \partial_M \downarrow & & \downarrow \partial_N \\ M^* & \xrightarrow{i^*} & N^* \end{array}$$

□

PROPOSITION 4.4. *Let (M, φ) be a t -torsion space and L an $A[t]$ -submodule of M . If M/L is projective over A , then $L = L^{\perp\perp}$ and L^\perp is a direct factor of M as an A -module.*

Proof. First observe that as an A -module L is finitely generated and projective. Let $i: L \rightarrow M$ be the natural injection. By Lemma 4.3 the map $i^\sharp \circ \varphi$ is surjective, thus the sequence

$$0 \longrightarrow L^\perp \xrightarrow{j} M \xrightarrow{i^\sharp \circ \varphi} L^\sharp \longrightarrow 0$$

is exact. Hence L^\perp is a direct factor of M as an A -module; in particular it is A -projective. Identifying L with $L^{\sharp\sharp}$ we can write the dual sequence as

$$0 \longrightarrow L \xrightarrow{i} M \xrightarrow{j^\sharp \circ \varphi^\sharp} (L^\perp)^\sharp \longrightarrow 0.$$

Notice that it is exact by Lemma 4.3. Again by Lemma 4.3 the sequence

$$0 \longrightarrow L^{\perp\perp} \longrightarrow M \xrightarrow{j^\sharp \circ \varphi^\sharp} (L^\perp)^\sharp \longrightarrow 0$$

is exact because L^\perp is a direct factor of M as an A -module. Since $\varphi^\sharp = \pm\varphi$, comparing the last two sequences we get the result. □

We now prove a fundamental result on the equivalence of t -torsion spaces.

THEOREM 4.5. *Let (M, φ) be an ϵ -hermitian t -torsion space and L a sublagrangian of (M, φ) . The quotient L^\perp/L carries a natural structure of t -torsion ϵ -hermitian space and its class in $W_{\text{tors}}(A[t])$ is the same as that of (M, φ) .*

Proof. We first prove the following lemma.

LEMMA 4.6. *Let (M, φ) be any ϵ -hermitian t -torsion space. The space $(M, \varphi) \perp (M, -\varphi)$ is metabolic.*

Proof of Lemma 4.6. We show that the image $L = \Delta(M)$ of the diagonal map $M \xrightarrow{\Delta} M \oplus M$ is a lagrangian. The condition $L \subseteq L^\perp$ is immediately verified. The quotient $(M \oplus M)/L$ is isomorphic to M , hence it is projective over A . It remains to see that $L^\perp \subseteq L$. If $(a, b) \in L^\perp$ we have $0 = \langle (a, b), (x, x) \rangle = \langle a - b, x \rangle$ for any $x \in M$. Since the pairing $\langle \cdot, \cdot \rangle$ is perfect, this implies $a = b$, i.e. $(a, b) \in L$. \square

We now prove the theorem. By Proposition 4.4, L^\perp is a direct factor of M as an A -module. Since $L \subseteq L^\perp$ is also a direct factor of M , the quotient L^\perp/L is projective. Denoting by \bar{a}, \bar{b} the classes modulo L of two elements $a, b \in L$, we define the hermitian structure of L^\perp/L by $\langle \bar{a}, \bar{b} \rangle = \langle a, b \rangle$. It is clear that $\langle a, b \rangle$ only depends on \bar{a} and \bar{b} . We first check that this pairing defines a t -torsion space. It is clearly ϵ -hermitian. The injectivity of the adjoint map $L^\perp/L \rightarrow (L^\perp/L)^\#$ follows immediately from Proposition 4.4. To show surjectivity consider any $A[t]$ -linear map $f: L^\perp \rightarrow A[t, t^{-1}]/A[t]$. Since L^\perp is a direct factor of M as an A -module, f , by Lemma 4.3, extends to an $A[t]$ -linear map $\tilde{f}: M \rightarrow A[t, t^{-1}]/A[t]$. Choose an $m \in M$ for which $\tilde{f} = \langle m, \cdot \rangle$. If \tilde{f} vanishes on L , then m is in L^\perp . This proves that L^\perp/L is a t -torsion space.

To show that L^\perp/L is equivalent to (M, φ) we check that the image of the diagonal map $\Delta: L^\perp \rightarrow M \oplus L^\perp/L$ is a lagrangian of $(M, -\varphi) \perp L^\perp/L$ which is, therefore, metabolic. It is easy to check that $\Delta(L^\perp)$ is contained in its own orthogonal. Conversely, if $(a, \bar{b}) \in M \oplus L^\perp/L$ is orthogonal to every (x, \bar{x}) , then $\langle a - b, x \rangle = 0$ for every $x \in L^\perp$. This means that $a - b$ is in $L^{\perp\perp}$, which by Proposition 4.4 coincides with L . We thus have $(a, \bar{b}) = (a, \bar{a}) \in \Delta(L^\perp)$. \square

The next proposition connects the Witt group of t -torsion spaces with the Witt group of A .

PROPOSITION 4.7. *The isomorphisms*

$$\partial_M: \text{Hom}_{A[t]}(M, A[t, t^{-1}]/A[t]) \rightarrow \text{Hom}_A(M, A)$$

induce a surjective homomorphism

$$\partial^W: W_{\text{tors}}(A[t]) \rightarrow W(A).$$

Proof. Associating to any t -torsion space (M, φ) the hermitian space $(M, \partial_M \circ \varphi)$ preserves isometries and orthogonal sums and, by Lemma 4.3, transforms metabolic t -torsion spaces into hyperbolic spaces (with the same lagrangian). Therefore it induces a homomorphism

$$\partial^W: W_{\text{tors}}(A[t]) \rightarrow W(A).$$

To find a preimage (M, φ) of a space (M, α) over A consider M as an $A[t]$ -module annihilated by t and replace $\alpha: M \rightarrow M^*$ by $\varphi = \partial_M^{-1} \circ \alpha$. \square

5. THE WITT GROUP OF EXTENDED SPACES

Let $W'(A[t, t^{-1}])$ be the group defined in the introduction.

THEOREM 5.1. *Let A be an associative ring with involution, in which 2 is invertible. The homomorphism*

$$\psi: W(A) \oplus W(A) \rightarrow W'(A[t, t^{-1}])$$

mapping (ξ, η) to $\xi + t\eta$ is an isomorphism.

Proof. The injectivity of ψ is based on the following result, whose proof will be given in §6.

PROPOSITION 5.2. *There exists a homomorphism*

$$\text{Res}: W'(A[t, t^{-1}]) \rightarrow W(A)$$

with the following properties:

R_1 : *For any constant space $\xi \in W(A) \subset W'(A[t, t^{-1}])$, $\text{Res}(\xi) = 0$.*

R_2 : *For any constant space $\xi \in W(A) \subset W'(A[t, t^{-1}])$, $\text{Res}(t \cdot \xi) = \xi$.*

Proof. See Theorem 6.7. \square