3. Small Flatspaces

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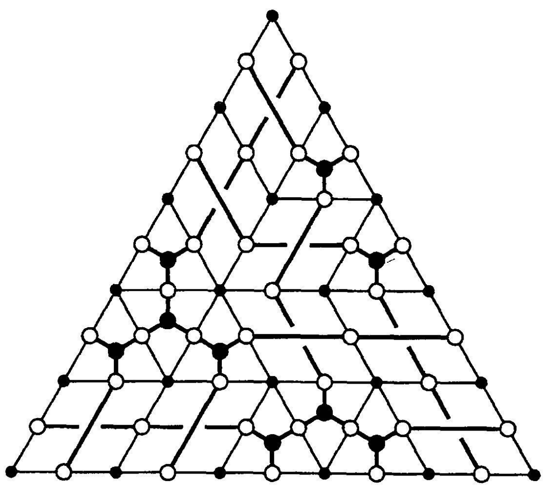
that two marked sides must meet in a 120° angle at the obtuse vertex on the border. However, the construction never introduces marked sides that meet in this angle.

We have established that the presence of any flatspace which is not a small triangle or rhombus gives rise to an increasable subset. This completes the proof. □

3. Small Flatspaces

Let h be a hive, all of whose flatspaces are small triangles or small rhombi. We construct a graph G from h as follows. G has one fat black vertex in the middle of each small triangle flatspace. In addition there is one circle vertex on every flatspace side. Each fat vertex is connected to the three vertices on the sides of its triangle, and the two circle vertices on opposite sides of a flat rhombus are connected. This graph is topologically equivalent to the reduced honeycomb tinkertoy of Knutson and Tao.

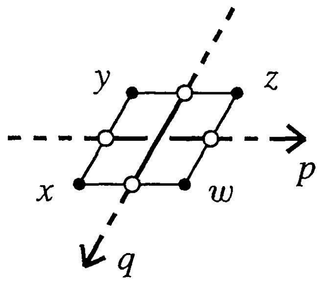


Lemma 1. *If h is a corner of its hive polytope $\rho^{-1}(\rho(h)) \cap C$, then G is acyclic.*

Proof. Suppose G has a non-trivial loop, and give this loop an orientation. Each hive vertex then has a well defined winding number, which is the number of times the loop goes around this vertex, counted positive in the counter clockwise direction. Note that the winding number is zero for each border vertex, and that some winding numbers are non-zero if the loop is not trivial.

For each $r \in \mathbf{R}$, let $h_r \in \mathbf{R}^H$ be the labeling which maps each hive vertex to the label of h at the vertex plus r times the winding number of the vertex. We will show that h_r is a hive for $r \in (-\epsilon, \epsilon)$, for a suitable $\epsilon > 0$. This implies that h is an interior point of a line segment contained in its hive polytope, which contradicts the assumption that h is a corner.

Choose any $\epsilon > 0$ such that each rhombus inequality that is strict for h is also satisfied for h_r when $|r| < \epsilon$. We claim that this ϵ will do. Consider any rhombus satisfied by h with equality:



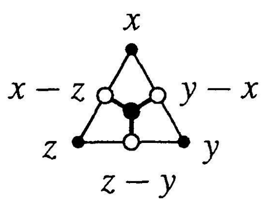
Suppose that the loop goes p times through the horizontal edge in the indicated direction and q times down through the other edge. Let the vertex with label x have winding number t. Then going clockwise around the rhombus, the winding numbers of the three other vertices are $t+p$, $t+p+q$, and $t+q$. It follows that the labels of h_r are

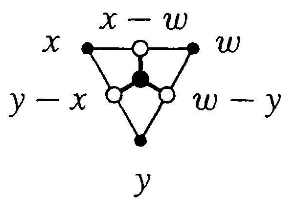
$$y' = y + r(t+p), \qquad z' = z + r(t+p+q),$$
$$x' = x + rt, \qquad w' = w + r(t+q).$$

Since the rhombus is flat for h, we have $x+z = y+w$. But this implies that $x' + z' = y' + w'$, and so the rhombus is also flat for h_r. □

PROPOSITION 2. *Let h be a hive which is a corner of its hive polytope $\rho^{-1}(\rho(h)) \cap C$. Suppose the flatspaces of h consist only of small triangles and small rhombi. Then the labels of h are integer linear combinations of the border labels.*

Proof. By Lemma 1, the graph G for h is acyclic. Label each circle vertex with the difference of the labels of the hive vertices on its side as shown below. A circle vertex on a horizontal side is always assigned the label of the left hive vertex minus the label of the right hive vertex on its side, etc.





By construction, the sum of the labels of three circle vertices surrounding any fat vertex is zero. Furthermore, if two circle vertices are connected by a single edge, then their labels are equal. This follows because the rhombus that separates them is satisfied with equality. We claim that all circle vertex labels are $\mathbf{Z}$-linear combinations of the border labels. Since this implies that also all labels of hive vertices are such linear combinations, this will finish the proof.

If the claim is false, let S be the non-empty set of circle vertices whose labels are not $\mathbf{Z}$-linear combinations of border labels, together with all fat vertices connected directly to one of these circle vertices. Since G is acyclic, some vertex $u \in S$ is connected to at most one other vertex in S.

Suppose u is a circle vertex. Then u can't be on the border of the big hive triangle, since its label would then be the difference of two border vertices, and so a $\mathbf{Z}$-linear combination of these. Therefore u is not an endpoint of G, so it is connected to a vertex v outside S. Since S contains all fat vertices connected to u by construction, v must be a circle vertex whose label is a $\mathbf{Z}$-linear combination of border labels. But u has the same label, a contradiction.

Therefore u must be a fat label, and exactly one of its three surrounding circle vertices is in S. This means that the labels of the other two circle neighbors are $\mathbf{Z}$-linear combinations of border labels. But since the sum of the labels of all three circle vertices surrounding u is zero, all three labels must be $\mathbf{Z}$-linear combinations of the border labels. This contradiction shows that S is empty, which concludes the proof. □

4. Proof of the Saturation Conjecture

We will call a functional on $\mathbf{R}^{H-B}$ *generic* if it takes its maximum at a unique point in $\rho^{-1}(b) \cap C$ for each $b \in \rho(C)$. It follows from the existence of secondary fans in linear programming [11, § 1] that the generic functionals form a dense open subset of $(\mathbf{R}^{H-B})^*$. We can now complete the proof of the saturation conjecture.

THEOREM 2. *Let* $(\lambda, \mu, \nu) \in \mathbf{Z}^{3n}$ *and* $N > 0$. *Then* $(\lambda, \mu, \nu) \in T_n$ *if and only if* $(N\lambda, N\mu, N\nu) \in T_n$.

Proof. As already noted, it is enough to show that if $b \in \rho(C) \cap \mathbf{Z}^B$ then the fiber $\rho^{-1}(b) \cap C$ contains an integral hive.