4. Proof of the saturation conjecture

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

By construction, the sum of the labels of three circle vertices surrounding any fat vertex is zero. Furthermore, if two circle vertices are connected by a single edge, then their labels are equal. This follows because the rhombus that separates them is satisfied with equality. We claim that all circle vertex labels are $\mathbf{Z}$-linear combinations of the border labels. Since this implies that also all labels of hive vertices are such linear combinations, this will finish the proof.

If the claim is false, let S be the non-empty set of circle vertices whose labels are not $\mathbf{Z}$-linear combinations of border labels, together with all fat vertices connected directly to one of these circle vertices. Since G is acyclic, some vertex $u \in S$ is connected to at most one other vertex in S.

Suppose u is a circle vertex. Then u can't be on the border of the big hive triangle, since its label would then be the difference of two border vertices, and so a $\mathbf{Z}$-linear combination of these. Therefore u is not an endpoint of G, so it is connected to a vertex v outside S. Since S contains all fat vertices connected to u by construction, v must be a circle vertex whose label is a $\mathbf{Z}$-linear combination of border labels. But u has the same label, a contradiction.

Therefore u must be a fat label, and exactly one of its three surrounding circle vertices is in S. This means that the labels of the other two circle neighbors are $\mathbf{Z}$-linear combinations of border labels. But since the sum of the labels of all three circle vertices surrounding u is zero, all three labels must be $\mathbf{Z}$-linear combinations of the border labels. This contradiction shows that S is empty, which concludes the proof. □

4. Proof of the Saturation Conjecture

We will call a functional on $\mathbf{R}^{H-B}$ *generic* if it takes its maximum at a unique point in $\rho^{-1}(b) \cap C$ for each $b \in \rho(C)$. It follows from the existence of secondary fans in linear programming [11, § 1] that the generic functionals form a dense open subset of $(\mathbf{R}^{H-B})^*$. We can now complete the proof of the saturation conjecture.

THEOREM 2. *Let* $(\lambda, \mu, \nu) \in \mathbf{Z}^{3n}$ *and* $N > 0$. *Then* $(\lambda, \mu, \nu) \in T_n$ *if and only if* $(N\lambda, N\mu, N\nu) \in T_n$.

Proof. As already noted, it is enough to show that if $b \in \rho(C) \cap \mathbf{Z}^B$ then the fiber $\rho^{-1}(b) \cap C$ contains an integral hive.

Fix a generic functional ω on $\mathbf{R}^{H-B}$ which maps a hive to a linear combination with positive coefficients of the labels at non-border hive vertices. For each $b \in \rho(C)$, let $\ell(b)$ be the unique hive in $\rho^{-1}(b) \cap C$ where ω is maximal. Then $\ell\colon \rho(C) \to C$ is a continuous piece-wise linear map [11, § 1]. Notice that since w has positive coefficients, $\ell(b)$ has no increasable subsets.

We want to prove that the labels of $\ell(b)$ are $\mathbf{Z}$-linear combinations of the labels of b. In particular $\ell(b)$ is an integral hive if b is integral. For a regular border $b \in \rho(C)$, Proposition 1 implies that the flatspaces of $\ell(b)$ consist of small triangles and rhombi; by Proposition 2 this implies that all labels of $\ell(b)$ are $\mathbf{Z}$-linear combinations of the labels of b. Finally, since the regular borders are dense in each maximal subcone of $\rho(C)$ where ℓ is linear, ℓ must be integrally defined everywhere. □

5. Remarks and questions

Knutson and Tao's proof of the saturation conjecture implies that Klyachko's inequalities for T_n can be produced by a simple recursive algorithm, which uses the inequalities for T_k, $1 \leq k \leq n-1$ ([9], [10], [12], [6]). A triple of partitions (λ, μ, ν) with $|\nu| = |\lambda| + |\mu|$ is in T_n if and only if

$$\sum_{i=1}^{k} \nu_{\gamma_i+k+1-i} \leq \sum_{i=1}^{k} \lambda_{\alpha_i+k+1-i} + \sum_{i=1}^{k} \mu_{\beta_i+k+1-i}$$

for all triples $(\alpha, \beta, \gamma) \in T_k$ with $\gamma_1 \leq n-k$. Another important consequence is Horn's conjecture, which says that the same inequalities describe which sets of eigenvalues can arise from two Hermitian matrices and their sum [8].

P. Belkale has shown that the inequality produced by a triple (α, β, γ) with Littlewood-Richardson coefficient $c^{\gamma}_{\alpha\beta} \geq 2$ follows from the other inequalities. Knutson, Tao, and Woodward have announced a proof that the remaining inequalities are independent, i.e. they describe the facets of the cone $\rho(C)$. Their proof uses an interesting operation of overlaying two hives, which is defined in terms of Knutson and Tao's honeycomb model [10].

These results have made it very interesting to determine which triples (λ, μ, ν) have coefficient $c^{\nu}_{\lambda\mu}$ equal to one. Fulton has conjectured that this is equivalent to $c^{N\nu}_{N\lambda, N\mu}$ being one for any $N \in \mathbf{N}$. This has been verified in all cases with $N|\nu| \leq 68$. (Recently Knutson and Tao have reported that they can prove this as well.)

For $n = 3$ it is easy to show that a triple of partitions has Littlewood-Richardson coefficient one if and only if it corresponds to a point on the