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Fix a generic functional  $\omega$  on  $\mathbf{R}^{H-B}$  which maps a hive to a linear combination with positive coefficients of the labels at non-border hive vertices. For each  $b \in \rho(C)$ , let  $\ell(b)$  be the unique hive in  $\rho^{-1}(b) \cap C$  where  $\omega$  is maximal. Then  $\ell: \rho(C) \rightarrow C$  is a continuous piece-wise linear map [11, §1]. Notice that since  $w$  has positive coefficients,  $\ell(b)$  has no increasable subsets.

We want to prove that the labels of  $\ell(b)$  are  $\mathbf{Z}$ -linear combinations of the labels of  $b$ . In particular  $\ell(b)$  is an integral hive if  $b$  is integral. For a regular border  $b \in \rho(C)$ , Proposition 1 implies that the flatspaces of  $\ell(b)$  consist of small triangles and rhombi; by Proposition 2 this implies that all labels of  $\ell(b)$  are  $\mathbf{Z}$ -linear combinations of the labels of  $b$ . Finally, since the regular borders are dense in each maximal subcone of  $\rho(C)$  where  $\ell$  is linear,  $\ell$  must be integrally defined everywhere.  $\square$

## 5. REMARKS AND QUESTIONS

Knutson and Tao's proof of the saturation conjecture implies that Klyachko's inequalities for  $T_n$  can be produced by a simple recursive algorithm, which uses the inequalities for  $T_k$ ,  $1 \leq k \leq n-1$  ([9], [10], [12], [6]). A triple of partitions  $(\lambda, \mu, \nu)$  with  $|\nu| = |\lambda| + |\mu|$  is in  $T_n$  if and only if

$$\sum_{i=1}^k \nu_{\gamma_i+k+1-i} \leq \sum_{i=1}^k \lambda_{\alpha_i+k+1-i} + \sum_{i=1}^k \mu_{\beta_i+k+1-i}$$

for all triples  $(\alpha, \beta, \gamma) \in T_k$  with  $\gamma_1 \leq n-k$ . Another important consequence is Horn's conjecture, which says that the same inequalities describe which sets of eigenvalues can arise from two Hermitian matrices and their sum [8].

P. Belkale has shown that the inequality produced by a triple  $(\alpha, \beta, \gamma)$  with Littlewood-Richardson coefficient  $c_{\alpha\beta}^{\gamma} \geq 2$  follows from the other inequalities. Knutson, Tao, and Woodward have announced a proof that the remaining inequalities are independent, i.e. they describe the facets of the cone  $\rho(C)$ . Their proof uses an interesting operation of overlaying two hives, which is defined in terms of Knutson and Tao's honeycomb model [10].

These results have made it very interesting to determine which triples  $(\lambda, \mu, \nu)$  have coefficient  $c_{\lambda\mu}^{\nu}$  equal to one. Fulton has conjectured that this is equivalent to  $c_{N\lambda, N\mu}^{N\nu}$  being one for any  $N \in \mathbf{N}$ . This has been verified in all cases with  $N|\nu| \leq 68$ . (Recently Knutson and Tao have reported that they can prove this as well.)

For  $n = 3$  it is easy to show that a triple of partitions has Littlewood-Richardson coefficient one if and only if it corresponds to a point on the

boundary of the cone  $\rho(C)$ . In general, Fulton's conjecture implies that the triples with coefficient one are exactly those corresponding to points in a collection of faces of  $\rho(C)$ . For  $n \geq 3$  this means that all triples corresponding to interior points in  $\rho(C)$  have coefficient at least two.

One approach for proving Fulton's conjecture is to show that if  $b \in \rho(C) \cap \mathbf{Z}^B$ , then any generic positive functional  $\omega$  on  $\mathbf{R}^{H-B}$  must be minimized (as well as maximized) at an integral hive in  $\rho^{-1}(b) \cap C$ . In fact, by Proposition 2 it is enough to prove:

*If  $b \in \rho(C)$  is a generic border and if a generic positive functional  $\omega$  is minimized at  $h \in \rho^{-1}(b) \cap C$ , then the flatspaces of  $h$  consist of small triangles and rhombi.*

Part of proving this is to specify when a border  $b$  is generic. We believe the statement is true if  $b$  avoids finitely many hyperplanes in  $\mathbf{R}^B$ .

The Littlewood-Richardson coefficients  $c_{\lambda\mu}^\nu$  have the following natural generalization. Given decreasing sequences of integers  $\nu$ , and  $\lambda(1), \dots, \lambda(r)$ , let  $c_{\lambda(1), \dots, \lambda(r)}^\nu$  denote the multiplicity of  $V_\nu$  in the holomorphic representation  $V_{\lambda(1)} \otimes \dots \otimes V_{\lambda(r)}$ . When  $\nu = (0, \dots, 0)$ , this specializes to the symmetric Littlewood-Richardson coefficient  $c_{\lambda(1), \dots, \lambda(r)}$  which is the dimension of the  $\mathrm{GL}_n(\mathbf{C})$ -invariant subspace of  $V_{\lambda(1)} \otimes \dots \otimes V_{\lambda(r)}$ . Postnikov and Zelevinsky have pointed out that the saturation conjecture as stated in the introduction implies a similar result for these generalized coefficients, i.e.

$$(5.1) \quad c_{\lambda(1), \dots, \lambda(r)}^\nu \neq 0 \iff c_{N\lambda(1), \dots, N\lambda(r)}^{N\nu} \neq 0.$$

Knutson has shown us that, by combining several hive triangles, one obtains a polytope whose integral points count these more general coefficients. This gives rise to another proof of (5.1).

In [3] other generalized Littlewood-Richardson coefficients related to quiver varieties are described. A different generalization related to Hecke algebras is defined in [7], and quantum Littlewood-Richardson coefficients are studied in [2]. It would be very interesting if these coefficients can be realized as the number of integral points in some polytopes.