7. EXPLICIT COMPUTATIONS AND CUBIC TRACE FORMS

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With this notation we have a commutative square

$$(32) \qquad \begin{array}{ccc} H^1_{\mathrm{fl}}(X, \mu_3) & \xrightarrow{i_*} & \mathrm{Pic}\,(C)[3] \\ {\scriptstyle f}\downarrow & & \uparrow{\scriptstyle j} \\ \mathcal{S}(C) & \xrightarrow{e'} & H(C)[3] \end{array}$$

where $j: H(C) \to \mathrm{Pic}\,(C)$ is the natural homomorphism $[M, q, N] \mapsto [M]$. Kneser [11, §6] has shown that j is an isomorphism (see also Section 2), so the two vertical maps in (32) are bijections and the horizontal maps are surjections.

Note that because of the exact sequence (31), the fibers of e' are in one-to-one correspondence with the elements of the group $C^\times/C^{\times 3}$. This is, of course, equivalent to Theorem 5.2, Part (ii).

7. Explicit computations and cubic trace forms

In this section we assume that $A := C \otimes K$ is a quadratic étale algebra over K. In this case the trace form $(x, y) \to \mathrm{Tr}_{A/K}(xy)$ is nondegenerate and gives rise to a natural isomorphism between the codifferent

$$C' = \{x \in A : \mathrm{Tr}_{A/K}(xC) \subset R\}$$

and the dual C^*. If M is a fractional C-ideal with $M^3 \simeq C'$, then, by Theorem 5.1, the cubic forms on M with primitive determining form are given by

$$(33) \qquad F_u(\mathbf{x}) = \mathrm{Tr}_{A/K}(ua\mathbf{x}^3),$$

where $a \in A$ is a fixed element with $aM^3 = C'$, and u is a unit of C. Moreover, by Theorem 5.1, two such forms F_u and F_v are C-isomorphic if and only if u and v represent the same element of $C^\times/(C^\times)^3$.

We shall compute explicitly some examples for $R = \mathbf{Z}$ using (33). In this case we have $C = \mathbf{Z}[t]/(f(t))$, where f is a monic degree-two polynomial with distinct roots and coefficients in $\mathbf{Z}$.

Let ω be the class of t in C. It is well-known, and easy to prove, that the codifferent C' is a principal fractional C-ideal generated by $f'(\omega)^{-1}$, where f' is the derivative of f. Hence, $[C^*]$ is trivial in $\mathrm{Pic}\,(C)$ (note that this holds more generally provided $\mathrm{Pic}\,(R) = 0$).

EXAMPLE 7.1. Let $C = \mathbf{Z}[\frac{1+\sqrt{-23}}{2}]$ (note that 23 is the smallest square-free positive integer N such that $A = \mathbf{Q}(\sqrt{-N})$ has class number divisible by 3; in fact $\mathrm{Pic}(C) \simeq \mathbf{Z}/3\mathbf{Z}$ (see [2]). The class group $\mathrm{Pic}(C)$ is generated by the class of

$$M = 2\mathbf{Z} + \omega\mathbf{Z},$$

where $\omega = \frac{1+\sqrt{-23}}{2}$. Thus the three classes of $\mathrm{Pic}(C)$ are represented by the ideals C, M and $\overline{M}$. The quadratic forms attached to C, M and $\overline{M}$ are respectively

$$x_1^2 + x_1x_2 + 6x_2^2, \quad 2x_1^2 + x_1x_2 + 3x_2^2, \quad 2x_1^2 - x_1x_2 + 3x_2^2.$$

One verifies also that $\theta = \omega - 2$ satisfies $M^3 = \theta C$, thus $(1/\theta\sqrt{-23})M^3 = C'$. Hence, by (33), the cubic C-form on M is given by

$$F(\mathbf{x}) = \mathrm{Tr}\left(\frac{\mathbf{x}^3}{\theta\sqrt{-23}}\right),$$

where $\mathbf{x} = 2x_1 + x_2\omega$. Similar computations can be done for $\overline{M}$ (taking $\theta = -1-\omega$ and the $\mathbf{Z}$-basis $\{2, -1+\omega\}$) and for C (with the basis $\{1, \omega\}$). The following table summarizes the results of these computations:

Module	Cubic Form	Determining Form
M	$-x_1^3 - 3x_1^2x_2 + 3x_1x_2^2 + 2x_2^3$	$2x_1^2 + x_1x_2 + 3x_2^2$
$\overline{M}$	$x_1^3 - 3x_1^2x_2 - 3x_1x_2^2 + 2x_2^3$	$2x_1^2 - x_1x_2 + 3x_2^2$
C	$x_2(3x_1^2 + 3x_1x_2 - 5x_2^2)$	$x_1^2 + x_1x_2 + 6x_2^2$

EXAMPLE 7.2. Let $C = \mathbf{Z}[\sqrt{79}]$. Here also $\mathrm{Pic}(C) \simeq \mathbf{Z}/3\mathbf{Z}$ (see [2]) (in fact 79 is the smallest square-free positive integer N such that $\mathbf{Q}(\sqrt{N})$ has class number divisible by 3).

The class group $\mathrm{Pic}(C)$ is generated by the class of

$$M = 9\mathbf{Z} + (4 + \sqrt{79})\mathbf{Z}.$$

Thus the three classes of $\mathrm{Pic}(C)$ are represented by the ideals C, M and $\overline{M}$. One verifies also that $\alpha = 52 - 5\sqrt{79}$ satisfies $M^3 = \alpha C$, thus $(1/2\alpha\sqrt{79})M^3 = C'$. The fundamental unit of C is $\tau = 80 + 9\sqrt{79}$; hence, by (33), the three nonisomorphic cubic C-forms on M are given by

$$F_{\tau^k}(\mathbf{x}) = \mathrm{Tr}\left(\frac{\tau^k}{2\alpha\sqrt{79}}\mathbf{x}^3\right),$$

where $\mathbf{x} = 9x_1 + (4+\sqrt{79})x_2$ and $k = -1, 0, 1$. Similar computations can be done for $\overline{M}$ (taking the $\mathbf{Z}$-basis $\{9, -4+\sqrt{79}\}$) and C (with the natural basis $\{1, \sqrt{79}\}$).

Module	Cubic Forms	Determining Form
M	$-68x_1^3 + 111x_1^2x_2 - 60x_1x_2^2 + 11x_2^3$ $5x_1^3 + 24x_1^2x_2 + 33x_1x_2^2 + 16x_2^3$ $868x_1^3 + 3729x_1^2x_2 + 5340x_1x_2^2 + 2549x_2^3$	$9x_1^2 + 8x_1x_2 - 7x_2^2$
$\overline{M}$	$-868x_1^3 + 3729x_1^2x_2 - 5340x_1x_2^2 + 2549x_2^3$ $-5x_1^3 + 24x_1^2x_2 - 33x_1x_2^2 + 16x_2^3$ $68x_1^3 + 111x_1^2x_2 + 60x_1x_2^2 + 11x_2^3$	$9x_1^2 - 8x_1x_2 - 7x_2^2$
C	$-9x_1^3 + 240x_1^2x_2 - 2133x_1x_2^2 + 6320x_2^3$ $3x_1^2x_2 + 79x_2^3$ $9x_1^3 + 240x_1^2x_2 + 2133x_1x_2^2 + 6320x_2^3$	$x_1^2 - 79x_2^2$

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