GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS

Autor(en): **BAUM, Paul / CONNES, Alain**

Objekttyp: **Article**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **46 (2000)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **13.09.2024**

Persistenter Link: https://doi.org/10.5169/seals-64793

L'Enseignement Mathématique, t. 46 (2000), p. 3–42

GEOMETRIC K-THEORY FOR LIE GROUPS AND FOLIATIONS

by Paul BAUM and Alain CONNES *)

1. INTRODUCTION

For a C^*-algebra A, let $K_0(A)$, $K_1(A)$ be its K-theory groups. Thus $K_0(A)$ is the algebraic K_0-theory of the ring A and $K_1(A)$ is the algebraic K_0-theory of the ring $A \otimes C_0(\mathbf{R}) = C_0(\mathbf{R}, A)$. If $A \to B$ is a morphism of C^*-algebras, then there are induced homomorphisms of abelian groups $K_i(A) \to K_i(B)$. Bott periodicity provides a six term K-theory exact sequence for each exact sequence $0 \to J \to A \to B \to 0$ of C^*-algebras.

Discrete groups, Lie groups, group actions and foliations give rise through their convolution algebra to a canonical C^*-algebra, and hence to K-theory groups. The analytical meaning of these K-theory groups is clear as a receptacle for indices of elliptic operators. However, these groups are difficult to compute. For instance, in the case of semi-simple Lie groups, the free abelian group with one generator for each irreducible discrete series representation is contained in $K_0 C^*G$ where C^*G is the reduced C^*-algebra of G. Thus an explicit determination of the K-theory in this case in particular involves an enumeration of the discrete series.

In this note we shall introduce a geometrically defined K-theory which specializes to discrete groups, Lie groups, group actions, and foliations. Its main features are its computability and the simplicity of its definition. In the case of semi-simple Lie groups it elucidates the role of the homogeneous space G/K (K the maximal compact subgroup of G) in the Atiyah-Schmid geometric construction of the discrete series [4]. Using elliptic operators we construct a natural map from our geometrically defined K-theory groups to the

*) The present paper was written in 1982 and distributed as an IHES preprint, but never published. We are grateful to the Editors for offering us to publish it without change in this Journal.

above analytic (i.e. C^*-algebra) K-theory groups. In all computed examples this map is an isomorphism. The picture that emerges is of two parallel theories: one analytic and one geometric. Elliptic operators provide a map from the geometric to the analytic theory. We give evidence for the conjecture that this map is always an isomorphism. In particular we prove that the map is injective for foliations with negatively curved leaves. We then explore some corollaries of this isomorphism conjecture. The injectivity is related through the work of G. G. Kasparov and A. S. Miscenko to the Novikov higher signature problem. We show how this problem leads to a conjecture on the invariance of certain foliation characteristic classes under leaf-wise homotopy equivalence. The surjectivity is related to a number of well-known C^*-algebra problems, such as the non-existence of idempotents in the reduced C^*-algebra of any torsion-free discrete group.

2. Lie group actions

G denotes a Lie group and X denotes a C^∞-manifold without boundary. Both G and X are assumed to be Hausdorff and second countable. G and X may have countably many connected components. G may be a countable discrete group.

DEFINITION 1. A C^∞ (right) action $X \times G \to X$ of G on X is *proper* if the map $X \times G \to X \times X$ given by

$$(x, g) \mapsto (x, xg)$$

is proper (i.e. the inverse image of any compact set is compact).

TERMINOLOGY. A *G-manifold* is a C^∞-manifold with a given (right) C^∞ G-action. If X, Y are G-manifolds a *G-map* from X to Y is a C^∞ G-equivariant map $f: X \to Y$. A G-manifold X is *proper* if the action of G on X is proper. A subset Δ of a proper G-manifold is *G-compact* if the image of Δ in the quotient space X/G is compact. A *G-vector bundle* on a G-manifold X is a C^∞-vector bundle E on X such that E is itself a G-manifold, the projection $E \to X$ is a G-map, and for each $(x, g) \in X \times G$ the map $E_x \to E_{xg}$ given by

$$u \mapsto ug \qquad (u \in E_x)$$

is linear.

A *G-vector bundle with G-compact support* on a proper G-manifold X is a triple (E_0, E_1, σ) where E_0, E_1 are G-vector bundles on X, $\sigma\colon E_0 \to E_1$ is a G-map which is linear on each fibre and Support(σ) is G-compact, where

$$\text{Support}(\sigma) = \{x \in X \mid \sigma\colon E_{0x} \to E_{1x} \text{ is not an isomorphism}\}.$$

For a G-manifold X, the analytic K-theory is the K-theory of the reduced crossed-product C^*-algebra $C_0(X) \rtimes G$. Here $C_0(X)$ is the C^*-algebra of all continuous complex-valued functions on X vanishing at infinity. We now proceed to define the geometric K-theory, denoted $K^*(X, G)$, and the natural map

$$K^i(X, G) \to K_i[C_0(X) \rtimes G] \qquad (i = 0, 1).$$

In doing this the G-manifold X will be "approximated" by proper G-manifolds. Note that the action of G on X is *not* required to be proper. Of special interest is the case when X is a point. For this case $C_0(\cdot) \rtimes G = C^*(G)$ where $C^*(G)$ is the reduced C^*-algebra of G.

Let Z be a proper G-manifold. $V^0_G(Z)$ denotes the collection of all complex G-vector bundles (E_0, E_1, σ) on Z with G-compact support. A group $K^0_G(Z)$ is defined by imposing on $V^0_G(Z)$ the same equivalence relation used by Atiyah-Segal ([5], [31])

$$K^0_G(Z) = V^0_G(Z)/\sim.$$

Addition in $K^0_G(Z)$ is given by direct sum $\xi \oplus \xi'$ of G-vector bundles with G-compact support.

To define $K^1_G(Z)$ let G act on $Z \times \mathbf{R}$ by:

$$(p, t)\, g = (pg, t)$$

($p \in Z$, $t \in \mathbf{R}$, $g \in G$). Set $V^1_G(Z) = V^0_G(Z \times \mathbf{R})$. Then

$$K^1_G(Z) = K^0_G(Z \times \mathbf{R}).$$

The basic properties of $K^*_G(Z)$ are stated and proved almost exactly as Atiyah-Segal did for compact G.

THOM ISOMORPHISM THEOREM. *On the proper G-manifold Z let E be an $\mathbf{R}$ G-vector bundle with a given G-invariant* Spin^c*-structure. Then*

$$K^*_G(Z) \cong K^*_G(E).$$

REMARK 2. The group $K^*_G(Z)$ is defined and used *only* for proper G-manifolds Z.

DEFINITION 3. Let X be a G-manifold. A *K-cocycle for* (X, G) is a triple (Z, ξ, f) such that

(1) Z is a proper G-manifold;

(2) $f: Z \to X$ is a G-map;

(3) $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$.

T^*Z is the cotangent bundle of Z and f^*T^*X is the pull-back to Z via f of T^*X. In (Z, ξ, f) all structures are C^∞ and G-equivariant.

The main result of this section is the construction of a canonical map μ from K-cocycles to the K-theory of the reduced crossed-product C^*-algebra $C_0(X) \rtimes G$.

THEOREM 4. *Each K-cocycle for (X, G) canonically determines an element in $K_*[C_0(X) \rtimes G]$.*

Outline of proof. First assume that $f: Z \to X$ is a submersion of Z onto X. Let τ be the cotangent bundle along the fibres of f. Using the Thom isomorphism theorem $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$ determines an element $\eta \in V_G^*(\tau)$. For $x \in X$, set $Z_x = f^{-1}(x)$. Then $\eta \in V_G^*(\tau)$ restricts to give $\eta_x \in V^*(T^*Z_x)$, which is the symbol of an elliptic operator on Z_x. Hence η is the symbol of a G-equivariant family D of elliptic operators, parametrized by the points of X. The K-theory index of D is the desired element of $K_*[C_0(X) \rtimes G]$:

$$\operatorname{Index}(D) \in K_*[C_0(X) \rtimes G].$$

If $f: Z \to X$ is not a submersion, then form the commutative diagram

$$\begin{array}{ccc} & & X \times Z \\ & {}^{i}\nearrow & \downarrow {}^{\rho} \\ Z & \xrightarrow[f]{} & X \end{array}$$

where $i(z) = (fz, z)$ and $\rho(x, z) = x$. Using the Thom isomorphism theorem, $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$ determines $\xi' \in V_G^*(T^*(X \times Z) \oplus \rho^*T^*X)$. The desired element of $K_*[C_0(X) \rtimes G]$ is then obtained as above from $(X \times Z, \xi', \rho)$. □

NOTATION. With D as in the proof of Theorem 4, $\operatorname{Index}(D) \in K_*[C_0(X) \rtimes G]$ will be denoted $\mu(Z, \xi, f)$. Observe that $\mu(Z, \xi, f)$ is the analytic index of the K-cocycle (Z, ξ, f). For $\xi \in V_G^i(T^*Z \oplus f^*T^*X)$, one has

$$\mu(Z, \xi, f) \in K_i[C_0(X) \rtimes G] \qquad (i = 0, 1).$$

Suppose given a commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{h} & Z_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & X & \end{array}$$

where Z_1, Z_2, X are G-manifolds with Z_1, Z_2 proper and f_1, f_2, h are G-maps. Using the Thom isomorphism theorem there is then a Gysin map

$$h_!: K^i_G(T^*Z_1 \oplus f_1^*T^*X) \to K^i_G(T^*Z_2 \oplus f_2^*T^*X) \qquad (i = 0, 1).$$

Just as for the ordinary analytic index of an elliptic operator on a G-manifold (with G compact) [6], the main property of the index μ is its invariance with respect to push-forward:

THEOREM 5. *The index map μ is compatible with Gysin maps in the following sense. If $\xi_1 \in V_G^*(T^*Z \oplus f_1^*T^*X)$, then $\mu(Z_1, \xi_1, f_1) = \mu(Z_2, h_!(\xi_1), f_2)$.*

REMARK 6. Theorems 4 and 5 indicate how to define the geometric K-theory $K^*(X, G)$ and the natural map

$$\mu: K^i(X, G) \to K_i[C_0(X) \rtimes G].$$

For a G-manifold X, let $\Gamma(X, G)$ be the collection of all K-cocycles (Z, ξ, f) for (X, G). On $\Gamma(X, G)$ impose the equivalence relation $\sim$, where $(Z, \xi, f) \sim (Z', \xi', f')$ if and only if there exists a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{h} & Z'' & \xleftarrow{h'} & Z' \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle f''} & \swarrow{\scriptstyle f'} & \\ & & X & & \end{array}$$

with $h_!(\xi) = h'_!(\xi')$.

DEFINITION 7. $K^*(X, G) = \Gamma(X, G)/\sim$.

The equivalence relation $\sim$ could also be defined as the equivalence relation generated by three elementary steps:

(1) cobordism;

(2) vector bundle modification;

(3) direct sum – disjoint union.

Addition in $K^*(X,G)$ is given by disjoint union of K-cocycles. Further,

$$K^*(X,G) = K^0(X,G) \oplus K^1(X,G),$$

where $K^i(X,G)$ is the subgroup of $K^*(X,G)$ determined by all K-cocycles (Z,ξ,f) with $\xi \in V_G^i(T^*Z \oplus f^*T^*X)$. The natural homomorphism of abelian groups

$$K^i(X,G) \to K_i[C_0(X) \rtimes G]$$

is defined by

$$(Z,\xi,f) \mapsto \mu(Z,\xi,f).$$

CONJECTURE. *For any G-manifold X, $\mu\colon K^i(X,G) \to K_i[C_0(X) \rtimes G]$ is an isomorphism.*

This conjecture is known to be true if X is a proper G-manifold. If X is proper there is a commutative diagram

$$\begin{array}{ccc} K^*(X,G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ & {\scriptstyle i_t}\searrow \quad \swarrow{\scriptstyle \alpha} & \\ & K_G^*(X) & \end{array}$$

in which each arrow is an isomorphism. $i_t\colon K^*(X,G) \to K_G^*(X)$ maps a K-cocycle (Z,ξ,f) to its topological index, and $\alpha \circ \mu\colon K^*(X,G) \to K_G^*(X)$ maps a K-cocycle (Z,ξ,f) to its analytic index. If G is compact then any G-manifold is proper and commutativity of the diagram is equivalent to the Atiyah-Singer index theorems of [6], [7], [8].

3. HOMOTOPY QUOTIENT

Let W be a topological space. $V^0(W)$ denotes the collection of all complex vector bundles (E_0, E_1, σ) on W with compact support. Thus E_0, E_1 are complex vector bundles on W and $\sigma\colon E_0 \to E_1$ is a morphism of complex vector bundles with Support(σ) compact, where

$$\text{Support}(\sigma) = \{p \in W \mid \sigma\colon E_{0p} \to E_{1p} \text{ is not an isomorphism}\}.$$

Also $V^1(W) = V^0(W \times \mathbf{R})$.

Suppose given an $\mathbf{R}$-vector bundle F on W. Following [9], a *twisted by F K-cycle* on W is a triple (M,ξ,ϕ) such that:

(1) M is a C^∞-manifold without boundary;

(2) $\phi\colon M \to W$ is a continuous map from M to W;

(3) $\xi \in V^*(T^*M \oplus \phi^*F)$.

As in [9] an equivalence relation is imposed on these twisted by F K-cycles to obtain the twisted by F K-homology of W:

$$K_*^F(W) = K_0^F(W) \oplus K_1^F(W).$$

$K_1^F(W)$ is the subgroup determined by all (M, ξ, ϕ) with $\xi \in V^i(T^*M \oplus \phi^*F)$. If F has a Spin^c-structure then $K_*^F(W)$ is isomorphic to $K_*(W)$, the K-homology of W.

With G as in §2 above, let EG be a contractible space on which G acts freely

$$EG \times G \to EG.$$

Given a G-manifold X, let G act on $EG \times X$ by

$$(p, x)\, g = (pg, xg)$$

($p \in EG$, $x \in X$, $g \in G$). The quotient space $[EG \times X]/G$ will be referred to as the homotopy quotient. Since T^*X is a G-vector bundle on X, the quotient $[EG \times T^*X]/G$ is a vector bundle on $[EG \times X]/G$. Denote this vector bundle by τ and consider the twisted by τ K-homology $K_*^\tau([EG \times X]/G)$. There is a map

$$K_*^\tau([EG \times X]/G) \to K^*(X, G).$$

This map is not quite canonical. First an orientation must be chosen for the Lie algebra of G, so assume that such an orientation has been chosen.

Let (M, ξ, ϕ) be a twisted by τ K-cycle on $[EG \times X]/G$. Now $EG \times X$ is the total space of a principal G-bundle over $[EG \times X]/G$ and this principal bundle can be pulled back via ϕ to yield a principal bundle Z over M

$$\begin{array}{ccc} EG \times X & \xleftarrow{\ \tilde{\phi}\ } & Z \\ \big\downarrow & & \big\downarrow{\scriptstyle \rho} \\ [EG \times X] & \xleftarrow[\ \phi\ ]{} & M. \end{array}$$

Let $\pi\colon EG \times X \to X$ be the projection and set $f = \pi \circ \tilde{\phi}$,

$$f\colon Z \to X.$$

$\xi \in V^*(T^*M \oplus \phi^*\tau)$ lifts to give $\tilde{\xi} \in V_G^*(\rho^*T^*M \oplus f^*T^*X)$. Denote the bundle along the fibres of $\rho\colon Z \to M$ by F. This is a trivial vector bundle since, for each $z \in Z$, F_z is canonically isomorphic to the Lie algebra of G. Using the orientation of this Lie algebra, F has a G-invariant Spin^c-structure so that $\xi \in V_G^*(\rho^*T^*M \oplus f^*T^*X)$ determines $\eta \in V_G^*(F \oplus \rho^*T^*M \oplus f^*T^*X)$. Now $F \oplus \rho^*T^*M = T^*Z$, so (Z, η, f) is a K-cocycle for (X, G). The map

$$K_*^\tau([EG \times X]/G) \to K^*(X, G)$$

is:

$$(M, \xi, \phi) \mapsto (Z, \eta, f).$$

This map has a dimension-shift in it. Set $\epsilon = \dim(G)$. Then with addition of indices mod 2 this map takes $K_i^\tau([EG \times X]/G)$ to $K^{i+\epsilon}(X, G)$.

LEMMA 1. *If G is torsion free then $K_*^\tau([EG \times X]/G) \to K^*(X, G)$ is an isomorphism.*

Proof. Let (Z, ξ, f) be a K-cocycle for (X, G). The action of G on Z is proper, so each isotropy group is compact. Since G is assumed to be torsion free this implies that the action of G on Z is free. Hence Z is a G-principal bundle over G/Z, and thus Z maps equivariantly to EG. Combining this with $f \colon Z \to X$ we obtain a commutative diagram

$$\begin{array}{ccc} EG \times X & \longleftarrow & Z \\ \downarrow & & \downarrow \rho \\ [EG \times X] & \longleftarrow & Z/G. \end{array}$$

Denote the map of Z/G to $[EG \times X]/G$ by ϕ. Then $\xi \in V_G^*(T^*Z \oplus f^*T^*X)$ determines $\xi' \in V_G^*(\rho^*T^*(Z/G) \oplus f^*T^*X)$. Since the action of G on Z is free ξ' descends to give $\theta \in V^*(T^*(Z/G) \oplus \tau)$. Then

$$(Z, \xi, f) \to (Z/G, \theta, \phi)$$

maps $K^*(X, G)$ to $K_*^\tau([EG \times X]/G)$ and provides an inverse to the map $K_*^\tau([EG \times X]/G) \to K^*(X, G)$. $\square$

REMARK 2. If G is the trivial one-element group then the isomorphism of the lemma becomes

$$K_*^{T^*X}(X) \cong K^*(X).$$

If X is a Spin^c-manifold then $K_*^{T^*X}(X) \cong K_*(X)$, so that in this case the isomorphism of the lemma becomes the Poincaré duality isomorphism $K_*(X) \cong K^*(X)$.

When G has torsion, the map $K_*^T([EG \times X]/G) \to K^*(X,G)$ can fail to be an isomorphism. The simplest example of this is obtained by taking X to be a point and $G = \mathbf{Z}/2\mathbf{Z}$.

When G has torsion, $K_*^T([EG \times X]/G)$ appears to be only a first approximation to $K^*(X,G)$ and $K_*[C_0(X) \rtimes G]$. The key point is that when G has torsion, there will be proper G-manifolds on which the G-action is not free.

4. Solvable simply connected Lie groups

The conjecture stated in §2 above is verified for (connected) solvable simply connected Lie groups by

PROPOSITION 1. *Let G be a (connected) solvable simply connected Lie group, and let X be a G-manifold. Then there is a commutative diagram*

$$\begin{array}{ccc} K^*(X,G) & \xrightarrow{\mu} & K_*[C_0(X) \rtimes G] \\ \downarrow & & \downarrow \\ K^*(X) & \longrightarrow & K_*[C_0(X)] \end{array}$$

in which each arrow is an isomorphism.

The proof depends on

LEMMA 2. *Let G be a (connected) solvable simply connected Lie group, and let Z be a proper G-manifold. Then there exists a G-map from Z to G.*

Proof of Lemma 2. Since the action of G on Z is proper all isotropy groups are compact. G has no non-trivial compact subgroups, so the action of G on Z is free. Therefore Z is a principal G-bundle with base Z/G. As G is itself a contractible space on which G acts freely, there is a G-map from Z to G. □

Proof of Proposition 1. In the diagram of the proposition the right vertical arrow is the Thom isomorphism of [13]. The lower horizontal arrow is the standard isomorphism which is valid for any locally compact Hausdorff topological space.

To define the left vertical arrow the first step is to use the lemma to construct an isomorphism

(1) $$K^*(X,G) \to K_G^*(T^*[X \times G] \oplus \pi_1^* T^* X).$$

Here G acts on $X \times G$ by

$$(x, g_1)\, g = (xg, g_1 g)$$

and $\pi_1 : X \times G \to X$ is the projection.

If (Z, ξ, f) is a K-cocycle for (X, G) then according to the lemma there exists a G-map $\psi: Z \to G$. Define $h: Z \to X \times G$ by $h(z) = (f\,z, \psi z)$ so that there is the evident commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{h} & X \times G \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi_1} & \\ & X & \end{array}$$

The isomorphism (1) is

$$(Z, \xi, f) \to h_!(\xi).$$

Next, $T^*[X \times G] \oplus \pi_1^* T^* X$ has a G-invariant Spin^c-structure so by the Thom isomorphism theorem of §2, there is an isomorphism

(2) $$K_G^*(T^*[X \times G] \oplus \pi_1^* T^* X) \cong K_G^*(X \times G).$$

Finally, the action of G on $X \times G$ is free and has $[X \times G]/G = X$. This yields an isomorphism

(3) $$K_G^*(X \times G) \cong K^*(X).$$

Composing (1), (2), (3) gives the left vertical arrow of the proposition. □

REMARK 3. The two vertical arrows in the diagram of the proposition are not quite canonical. First an orientation must be chosen for the Lie algebra of G. There is no dimension shift in the horizontal arrows of the proposition. If $\epsilon = \dim(G)$, then the left vertical arrow maps $K^i(X, G)$ to $K^{i+\epsilon}(X)$, and the right vertical arrow maps $K_i[C_0(X) \rtimes G]$ to $K_{i+\epsilon}[C_0(X)]$.

5. THE GEOMETRIC K-THEORY FOR $\pi_0 G$ FINITE

In this section we shall determine the geometric group $K^*(X, G)$ whenever G has only a finite number of connected components. The main point is the existence of a final object (namely $H\backslash G$, where H is the maximal compact subgroup of G) in the category of proper G-manifolds.

Throughout this section G is a Lie group with a finite number of connected components. H denotes the maximal compact subgroup of G. And $\mathfrak{g}$, $\mathfrak{h}$ are the Lie algebras

$$0 \to \mathfrak{h} \to \mathfrak{g} \to \mathfrak{h}\backslash\mathfrak{g} \to 0.$$

Passing to dual spaces (over $\mathbf{R}$):

$$0 \leftarrow \mathfrak{h}^* \leftarrow \mathfrak{g}^* \leftarrow (\mathfrak{h}\backslash\mathfrak{g})^* \leftarrow 0.$$

By the co-adjoint representation H acts on $(\mathfrak{h}\backslash\mathfrak{g})^*$

$$(\mathfrak{h}\backslash\mathfrak{g})^* \times H \to (\mathfrak{h}\backslash\mathfrak{g})^*.$$

Given a G-manifold X, let H act on $X \times (\mathfrak{h}\backslash\mathfrak{g})^*$ by

$$(x, u)\, h = (xh, uh)$$

($x \in X$, $u \in (\mathfrak{h}\backslash\mathfrak{g})^*$, $h \in H$).

PROPOSITION 1. *For any G-manifold X there is a canonical isomorphism of abelian groups*

$$K_H^i(X \times (\mathfrak{h}\backslash\mathfrak{g})^*) \to K^i(X, G) \qquad (i = 0, 1).$$

REMARK 2. The isomorphism of the proposition is completely canonical and has no shift of dimension.

COROLLARY 3. *Set* $\epsilon = \dim(\mathfrak{h}\backslash\mathfrak{g})$. *If the co-adjoint action of H on* $(\mathfrak{h}\backslash\mathfrak{g})^*$ *is* Spin^c, *then*

$$K_H^i(X) \cong K^{i+\epsilon}(X, G).$$

Proof of Corollary 3. If the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin^c, then the Thom isomorphism [1] applies to give an isomorphism

$$K_H^i(X) \to K_H^{i+\epsilon}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*).$$

Composing this with the isomorphism of Proposition 1 proves the corollary. □

REMARK 4. Set $H\backslash G = \{Hg \mid g \in G\}$. There is the evident (right) action of G on $H\backslash G$

$$(H\backslash G) \times G \to H\backslash G.$$

The action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin^c if and only if $H\backslash G$ admits a G-invariant Spin^c-structure.

To analyze the case when the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is not Spin^c, fix an H-invariant Euclidean structure on $(\mathfrak{h}\backslash\mathfrak{g})^*$. Proceed as in [15]. Since H is connected, the co-adjoint representation maps H into $\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*$. Let $\mathrm{Spin}(\mathfrak{h}\backslash\mathfrak{g})^*$ be the non-trivial 2-fold covering of $\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*$ and form the commutative diagram

$$\begin{array}{ccc} \widetilde{H} & \longrightarrow & \mathrm{Spin}(\mathfrak{h}\backslash\mathfrak{g})^* \\ \downarrow & & \downarrow \\ H & \longrightarrow & \mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^* \end{array}$$

where $\widetilde{H} = H \times_{\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*} \mathrm{Spin}(\mathfrak{h}\backslash\mathfrak{g})^*$ is the 2-fold covering of H obtained by pulling-back the Spin covering of $\mathrm{SO}(\mathfrak{h}\backslash\mathfrak{g})^*$. There is then ([1]) the Thom isomorphism

$$K^i_{\widetilde{H}}(X) \to K^{i+\epsilon}_{\widetilde{H}}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*).$$

Moreover, let $u \in \widetilde{H}$ be the non-identity element of $\widetilde{H}$ which maps to the identity element of H by the projection $\widetilde{H} \to H$. If E is any $\widetilde{H}$-vector bundle on X, there is the direct sum decomposition

$$E = E_+ \oplus E_-$$

where $E_\pm = \{v \in E \mid vu = \pm v\}$. This leads to a direct sum decomposition of $K^*_{\widetilde{H}}(X)$:

$$K^i_{\widetilde{H}}(X) = \left[K^i_{\widetilde{H}}(X)\right]_+ \oplus \left[K^i_{\widetilde{H}}(X)\right]_- ,$$

where $\left[K^i_{\widetilde{H}}(X)\right]_\pm$ is obtained by only using $E_\pm$. Note that $\left[K^i_{\widetilde{H}}(X)\right]_+ \cong K^i_H(X)$.

COROLLARY 5. *For any G-manifold X, there is an isomorphism of abelian groups*

$$\left[K^i_{\widetilde{H}}(X)\right]_- \to K^{i+\epsilon}(X, G),$$

$i = 0, 1$, $\epsilon = \dim(\mathfrak{h}\backslash\mathfrak{g})$.

Proof. The Thom isomorphism

$$K^i_{\widetilde{H}}(X) \to K^{i+\epsilon}_{\widetilde{H}}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*)$$

gives an isomorphism

$$\left[K^i_{\tilde{H}}(X)\right]_- \to K_H^{i+\epsilon}(X \times (\mathfrak{h}\backslash\mathfrak{g})^*)\,.$$

Combining this with the isomorphism of Proposition 1 proves Corollary 5. □

The essential point in the proof of Proposition 1 is given by

LEMMA 6. *Let Z be any proper G-manifold. Then there exists a G-map from Z to $H\backslash G$.*

Proof. Assume for simplicity that $H\backslash G$ admits a G-invariant Riemannian metric of non-positive curvature. This is the case if G is semi-simple [17].

It follows easily from the slice theorem of Palais [23] that Z can be covered by open sets $U_0, U_1, U_2, \ldots$ such that each U_i is mapped into itself by G, $\{U_i\}$ is a locally finite cover of Z, and there exist G-maps $f_i\colon U_i \to H\backslash G$. Two points in $H\backslash G$ are joined by a unique geodesic. Let $\phi_0\colon U_0 \cup U_1 \to \mathbf{R}$, $\phi_1\colon U_0 \cup U_1 \to \mathbf{R}$ be a C^∞ partition of unity on $U_0 \cup U_1$ subordinate to the covering U_0, U_1 and with each ϕ_i constant on orbits. Then $\phi_0 f_0 + \phi_1 f_1$ is a G-map from $U_0 \cup U_1$ to $H\backslash G$ where $(\phi_0 f_0 + \phi_1 f_1)$ means the weighted average (by weights $\phi_0(x)$, $\phi_1(x)$) of $f_0(x), f_1(x)$ along the unique geodesic joining $f_0(x)$ and $f_1(x)$. Iterating this construction produces the desired G-map from Z to $H\backslash G$.

The general case has been proved by A. Borel [10]. □

Proof of Proposition 1. Let (Z, ξ, f) be a K-cocycle for (X, G). According to Lemma 6 there is a G-map $\theta\colon Z \to H\backslash G$. Let $h\colon Z \to X \times (H\backslash G)$ be

$$h(z) = (f\,z, \theta z)\,.$$

Form the evident commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ h\ } & X \times (H\backslash G) \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \pi} & \\ & X & \end{array}$$

where $\pi\colon X \times (H\backslash G) \to X$ is the projection.

Define an isomorphism

(1) $$K^i(X, G) \to K^i_G(T^*[X \times H\backslash G] \oplus \pi^* T^* X)$$

by

$$(Z, \xi, f) \to h_!(\xi)\,.$$

Now $T^*[X \times H\backslash G] \oplus \pi^*T^*X = \pi^*T^*X \oplus \pi^*T^*X \oplus \rho^*T^*(H\backslash G)$, where $\rho\colon X \times H\backslash G \to H\backslash G$ is the projection. $\pi^*T^*X \oplus \pi^*T^*X$ has a G-invariant Spin^c-structure. Hence the Thom isomorphism theorem applies to give an isomorphism

$$(2) \qquad K_G^i(T^*[X \times H\backslash G] \oplus \pi^*T^*X) \to K_G^i(\rho^*T^*(H\backslash G)).$$

Next, there is the identification

$$[X \times (\mathfrak{h}\backslash\mathfrak{g})^*] \times_H G = \rho^*T^*(H\backslash G).$$

This identification gives an induction isomorphism

$$(3) \qquad K_H^i[X \times (\mathfrak{h}\backslash\mathfrak{g})^*] \to K_G^i(\rho^*T^*(H\backslash G)).$$

Starting with an H-vector bundle E on $X \times (\mathfrak{h}\backslash\mathfrak{g})^*$ the induction isomorphism takes E to $E \times_H G$. Combining the isomorphisms (1), (2), (3) proves the proposition. □

REMARK 7. Of special interest is the case when X is a point. By the above proposition

$$K^{\epsilon}(\cdot, G) \cong R(\widetilde{H})_-$$
$$K^{1+\epsilon}(\cdot, G) = 0.$$

Here $\epsilon = \dim(\mathfrak{h}\backslash\mathfrak{g})$ and $R(\widetilde{H})_- = K^0_{\widetilde{H}}(\cdot)_-$ is the free abelian group with one generator for each irreducible representation of $\widetilde{H}$ which is *not* a representation of H. If the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin, then there is an identification $R(\widetilde{H})_- = R(H)$. The second-named author (A. Connes) and independently G. G. Kasparov [20] have conjectured that Dirac induction gives an isomorphism

$$K_{\epsilon}[C^*G] \cong R(\widetilde{H})_-$$
$$K_{1+\epsilon}[C^*G] = 0.$$

For connected complex semi-simple groups M. Pennington and R. Plymen [25], [28], have verified this conjecture. These results of M. Pennington and R. Plymen combined with the proposition of this section verify the isomorphism conjecture stated in §2 above in a number of interesting cases. Note that (due to the proposition of this section) the Connes-Kasparov conjecture on K_*C^*G is a special case of the isomorphism conjecture of §2.

Let G be a connected semi-simple Lie group with finite center. The lemma of this section elucidates the role of $H\backslash G$ in the Atiyah-Schmid geometric construction of the discrete series [4]. Atiyah and Schmid obtain the discrete series representations by using the Dirac operator on $H\backslash G$. As noted in the introduction $K_0[C^*G]$ contains a free abelian group with one generator for each (irreducible) discrete series representation. By the lemma, however, all of $K^*(\cdot, G)$ is obtained from $H\backslash G$. If (as conjectured in §2 above) $K^*(\cdot, G) \cong K_*(C^*G)$, then not only the discrete series, but *all* of $K_*(C^*G)$ can be obtained from $H\backslash G$.

At this juncture one might ask, "*Why not simply define* $K^i(X, G) = K^i_H(X)$ *?*" We believe that there are compelling reasons for not doing this. First, this misses the dimension-shift by $\epsilon = \dim(H\backslash G)$. Second, this overlooks the issue of whether or not the action of H on $(\mathfrak{h}\backslash\mathfrak{g})^*$ is Spin^c. Third, this greatly obscures the relation of K-theory to index theory. Finally, in the case of discrete groups and foliations there is no maximal compact subgroup so that if this were done there would be no unified theory for Lie groups, discrete groups, and foliations.

6. Discrete groups: Chern character

In this section G is a discrete group which is either finite or countable infinite. For a G-manifold X, $K^*(X, G)$ was defined in §2 above. As in §3 there is the natural map

$$K_*^\tau([EG \times X]/G) \to K^*(X, G),$$

where $\tau = [EG \times T^*X]/G$.

Proposition 1. *Let G be a discrete group and X a G-manifold. Then*

$$K_*^\tau([EG \times X]/G) \otimes_{\mathbf{Z}} \mathbf{Q} \to K^*(X, G) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is injective.

Remark 2. When X is a point, Proposition 1 asserts that

$$K_*(BG) \otimes_{\mathbf{Z}} \mathbf{Q} \to K^*(\cdot, G) \otimes_{\mathbf{Z}} \mathbf{Q}$$

is injective.

The proposition is proved by defining the *Chern character*. This is a natural map

$$K^*(X,G) \to H^{\tau}_*([EG \times X]/G; \mathbf{C}).$$

Here $H^{\tau}_*([EG \times X]/G; \mathbf{C})$ denotes the homology (with coefficients the complex numbers $\mathbf{C}$) of the pair $(B\tau, S\tau)$ where $B\tau, S\tau$ are the unit ball and unit sphere bundles of τ with respect to any continuous Euclidean structure chosen for τ

$$H^{\tau}_*([EG \times X]/G; \mathbf{C}) = H_*((B\tau, S\tau), \mathbf{C}).$$

The Chern character can be defined by the following five-step procedure, which is similar to a procedure used by M.F. Atiyah [2].

Step 1. Let (Z, ξ, f) be a K-cocycle for (X,G). Form the commutative diagram

$$\begin{array}{ccc} Z & \xrightarrow{\ h\ } & X \times Z \\ & {\scriptstyle f}\searrow \qquad \swarrow{\scriptstyle \pi_1} & \\ & X & \end{array}$$

where $h(z) = (fz, z)$ and $\pi_1(x,z) = x$. Consider $h_!(\xi) \in K^*_G(T^*(X \times Z) \oplus \pi_1^* T^*X)$. Now $T^*(X \times Z) = \pi_1^* T^*X \oplus \pi_2^* T^*Z$ where $\pi_2(x,z) = z$. Since $\pi_1^* T^*X \oplus \pi_1^* T^*X$ has a G-invariant Spin^c-structure, the Thom isomorphism theorem gives an isomorphism

$$K^*_G(T^*(X \times Z) \oplus \pi_1^* T^*X) \cong K^*_G(\pi_2^* T^*Z).$$

Via this isomorphism $h_!(\xi)$ determines $\xi' \in K^*_G(\pi_2^* T^*Z)$. Using G-invariant connections and the Chern-Weil curvature theory of characteristic classes, let ω be the differential form on $X \times T^*Z = \pi_2 T^*Z$ which represents the Atiyah-Singer answer for the index of a family of elliptic operators [7]. Thus ω is a G-invariant closed differential form with G-compact support which represents $\mathrm{ch}(\xi') \cup \pi_2^* \,\mathrm{Td}(\mathbf{C} \otimes_{\mathbf{R}} T^*Z)$. Here ch is the usual Chern character and Td is the Todd polynomial.

Step 2. The action of G on $X \times T^*Z = \pi_2^* T^*Z$ is proper. This implies that the quotient space $\pi_2^* T^*Z/G$ is a rational homology manifold. The differential form ω of Step 1 is closed, G-invariant and has G-compact support. Hence ω descends to determine a cohomology class $\underline{\omega}$, with compact support, on $\pi_2^* T^*Z/G$

$$\underline{\omega} \in H^*_c(\pi_2^* T^*Z/G; \mathbf{C}).$$

Step 3. On $X \times Z$ choose a G-invariant Euclidean structure for $\pi_1^* T^* X$ and let $B\pi_1^* T^* X$, $S\pi_1^* T^* X$ be the unit ball and unit sphere bundles. The rational homology manifold $T^*(X \times Z)/G$ is oriented. This gives a Poincaré duality isomorphism

$$H_c^*(\pi_2^* T^* Z/G; \mathbf{C}) \cong H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C}).$$

Using this isomorphism, $\underline{\omega} \in H_c^*(\pi_2^* T^* Z/G; \mathbf{C})$ determines

$$\mathrm{Dual}(\underline{\omega}) \in H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C}).$$

Step 4. On $[EG \times X \times Z]/G$ let $\widetilde{\tau}$ be the vector bundle $[EG \times \pi_1^* T^* X]/G$. Consider the evident map

$$\widetilde{\tau} = [EG \times \pi_1^* T^* X]/G \to \pi_1^* T^* X/G.$$

A typical fibre of this map is of the form $B\Gamma$ where Γ is an isotropy group for the action of G on $\pi_1^* T^* X$. Since this action is proper, Γ is a finite group and $H_i(B\Gamma; \mathbf{Q}) = 0$ for $i > 0$. Hence the map

$$\widetilde{\tau} \to \pi_1^* T^* X/G$$

is an isomorphism in rational homology. This gives an isomorphism:

$$H_*((B\widetilde{\tau}, S\widetilde{\tau}); \mathbf{C}) \cong H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C}).$$

By this isomorphism Dual $(\underline{\omega}) \in H_*((B\pi_1^* T^* X/G, S\pi_1^* T^* X/G); \mathbf{C})$ determines $\underline{\omega} \in H_*((B\widetilde{\tau}, S\widetilde{\tau}); \mathbf{C})$.

Step 5. The projection $\widetilde{\tau} = [EG \times T^* X \times Z]/G \to [EG \times T^* X]/G = \tau$ induces a map of homology

$$H_*((B\widetilde{\tau}, S\widetilde{\tau}); \mathbf{C}) \to H_*((B\tau, S\tau); \mathbf{C}).$$

The image of $\widetilde{\omega}$ under this map is, by definition, the Chern character of the original K-cocycle (Z, ξ, f).

Proof of Proposition 1. The following diagram is commutative

$$\begin{array}{ccc} K_*^{\tau}([EG \times X]/G) & \longrightarrow & K^*(X, G) \\ & \searrow \qquad \swarrow & \\ & H_*^{\tau}([EG \times X]/G; \mathbf{C}) & \end{array}$$

where the left vertical arrow is the usual K-homology Chern character [9]. Since the usual K-homology Chern character is rationally injective, this forces the horizontal arrow to be rationally injective. $\square$

REMARK 3. For G discrete the reduced C^*-algebra of G, denoted C^*G, comes equipped with a trace. An element in C^*G is a formal sum $\sum_{g\in G} \lambda_g g$ where $\lambda_g \in \mathbf{C}$. The trace of such an element is λ_1 where 1 is the identity element of G. This trace then induces a map

$$\mathrm{tr}\colon K_0\, C^*G \to \mathbf{R}.$$

Let Z be a proper G-manifold and let D be a G-invariant elliptic operator on Z. If ξ is the symbol of D then (Z,ξ) is a K-cocycle for $(\cdot, G)$ and the Chern character defined above assigns to (Z,ξ)

$$\mathrm{ch}(Z,\xi) \in H_*(BG;\mathbf{C}).$$

Let $\epsilon\colon BG \to \cdot$ be the map of BG to a point. Identify $H_*(\cdot, \mathbf{C}) = \mathbf{C}$ and consider

$$\epsilon_*\, \mathrm{ch}(Z,\xi) \in \mathbf{C}.$$

The K-theory index of the elliptic operator D is an element of $K_0\, C^*G$

$$\mathrm{Index}(D) \in K_0\, C^*G.$$

We then have the following formula for $\mathrm{tr}[\mathrm{Index}(D)]$:

$$\mathrm{tr}[\mathrm{Index}(D)] = \epsilon_*\, \mathrm{ch}(Z,\xi).$$

For the special case when the action of G on Z is free this formula was obtained by M.F. Atiyah [3].

7. COROLLARIES OF THE ISOMORPHISM CONJECTURE

The conjecture stated in §2 above asserts that

$$\mu\colon K^*(X,G) \to K_*[C_0(X) \rtimes G]$$

is an isomorphism. Suppose that G is a discrete group and X is a point. The conjecture then asserts that $\mu\colon K^*(\cdot, G) \to K_* C^*G$ is an isomorphism where C^*G is the reduced C^*-algebra of G. Throughout this section G will be a discrete group and we shall consider some corollaries of the conjecture that $\mu\colon K^0(\cdot, G) \to K_0\, C^*G$ is an isomorphism. "*Proof*" will mean "*Proof modulo the conjecture*".

COROLLARY 1. *If G is torsion free then* $\mathrm{tr}\colon K_0 C^*G \to \mathbf{R}$ *maps* $K_0 C^*G$ *onto the integers* $\mathbf{Z}$.

"Proof". Let (Z, ξ) be a K-cocycle for $(\cdot, G)$. Let D be a G-invariant elliptic operator on Z whose symbol is ξ. By the definition of $\mu\colon K^0(\cdot, G) \to K_0 C^*G$ given in §2 above

$$\mu(Z, \xi) = \mathrm{Index}(D).$$

If G is torsion free then the action of G on Z must be free. Hence Atiyah's result applies [3] and $\mathrm{tr}[\mathrm{Index}(D)]$ must be an integer. Thus the surjectivity of $\mu\colon K^0(\cdot, G) \to K_0 C^*G$ implies that $\mathrm{tr}\colon K_0 C^*G \to \mathbf{R}$ takes on only integer values. □

COROLLARY 2. *If G is torsion free then there are no non-trivial projections in C^*G.*

"Proof". A non-trivial projections in C^*G would give an element $\alpha \in K_0 C^*G$ with $0 < \mathrm{tr}(\alpha) < 1$. □

REMARK 3. For G torsion-free abelian, Corollary 2 can be proved by applying Pontrjagin duality. At the other extreme, Pimsner and Voiculescu [27] have proved that Corollary 2 is valid for a finitely generated free group.

In the statement of Corollary 2 it is essential that C^*G be the reduced C^*-algebra of G. Corollary 2 is not valid if one uses the maximal C^*-algebra $C^*_{\max}G$.

A classical conjecture [24] in the theory of group rings is that the group ring of a torsion-free group has no (non-trivial) divisors of zero. J. Cohen has observed that Corollaries 1 and 2 may be relevant to this zero-divisor conjecture.

If G has torsion then we conjecture that $\mathrm{tr}\colon K_0 C^*G \to \mathbf{R}$ maps $K_0 C^*G$ onto the additive subgroup of $\mathbf{Q}$ generated by all rational numbers of the form $\frac{1}{n}$, where n is the order of a finite subgroup of G. This would follow from the conjectured surjectivity of $K^0(\cdot, G) \to K_0 C^*G$ plus the unproved assertion that $\mathrm{tr}[\mathrm{Index}(D)]$ can only take on such values, where D is any G-invariant elliptic operator on a proper G-manifold.

COROLLARY 4. *The Novikov conjecture on homotopy invariance of higher signatures* [11].

"Proof". Let M be a closed oriented C^∞-manifold, $G = \pi_1(M)$ and let $f\colon M \to BG$ be the classifying map of the universal covering space of M.

The Novikov conjecture is that

$$\langle \mathbf{L}(M) \cup f^*(a), [M] \rangle$$

is an invariant of oriented homotopy type, where $\mathbf{L}(M)$ is the total $\mathbf{L}$ class of TM and a is any element in $H^*(BG; \mathbf{Q})$.

Kasparov [19] and Miscenko-Fomenko [21] [22] define a map

$$K_0(BG) \to K_0\, C^*G$$

and prove that the Novikov conjecture is implied by its rational injectivity. This enabled them to prove the Novikov conjecture for any discrete subgroup of a linear Lie group. The relation with our conjecture is clear from the following commutative diagram

$$\begin{array}{ccc} K_0(BG) & \longrightarrow & K_0\, C^*G \\ & \searrow \quad \swarrow & \\ & K^0(\cdot\,, G) & \end{array}$$

and the Proposition of § 6 above. (In this factorization, the topological definition of K-homology given in [9] is being used.) □

COROLLARY 5. *(Stable) Riemannian geometry conjectures of Gromov-Lawson-Rosenberg* [30].

For the same reason our conjecture implies the stable[1]) form of the Riemannian geometry conjectures of Gromov-Lawson-Rosenberg [30] on topological obstructions to the existence of metrics of positive scalar curvature.

8. TWISTING BY A 2-COCYCLE

This section is motivated by the papers [16], [26], [29], on the range of the trace for the C^*-algebra of the projective regular representation of a discrete group.

All of § 2 adapts to the projective situation where together with the G-manifold X one is given a 2-cocycle $\gamma \in Z^2(X \rtimes G, S^1)$. For simplicity we

[1]) Paul Baum comments: It is important to emphasize "stable" because Thomas Schick has shown that the original unstable Gromov-Lawson-Rosenberg conjecture is false. On the other hand, Stephan Stolz (with contributions from J Rosenberg and others) has proved that the real form of Baum-Connes implies the stable Gromov-Lawson-Rosenberg conjecture Also, Max Karoubi and I have proved that the usual (i e complex K-theory) form of Baum-Connes implies the real form of Baum-Connes

shall stick to the case $X = \mathrm{pt} = \cdot$ and G discrete $= \Gamma$; then $\gamma \in Z^2(\Gamma, S^1)$ is a map: $\Gamma \times \Gamma \to S^1$ such that:

$$\gamma(g_2, g_3)\,\gamma(g_1 g_2, g_3)^{-1}\,\gamma(g_1, g_2 g_3)\,\gamma(g_1, g_2)^{-1} = 1 \text{ for every } g_1, g_2, g_3 \in \Gamma .$$

Given a *proper* Γ-manifold Z, a (Γ,γ)-vector bundle on Z is a smooth (complex) vector bundle E on Z together with a smooth map $E \times \Gamma \to E$ such that (with $\pi\colon E \to Z$ the projection):

a) $\pi(\xi g) = \pi(\xi) g$ for each $\xi \in E$, $g \in \Gamma$;

b) $\xi(g_1 g_2) = \gamma(g_1, g_2)\,(\xi g_1) g_2$ for each $g_1, g_2 \in \Gamma$.

In b), $\gamma(g_1, g_2) \in S^1$ is viewed as a complex number of modulus 1. As in §2, we let $V^0_{(\Gamma,\gamma)}(Z)$ be the collection of triples (E_0, E_1, σ) where E_0, E_1 are (Γ,γ)-vector bundles over Z and σ is a smooth morphism of vector bundles such that:

1) $\sigma(\xi g) = \sigma(\xi) g$ for each $\xi \in E_0$, $g \in \Gamma$;

2) Support(σ) is Γ-compact.

The groups $K^i_{(\Gamma,\gamma)}(Z)$ are then defined as in [5], [31]. The Thom isomorphism as formulated in §2 still holds in this context, and this allows us to define Gysin maps:

$$h!\colon K^i_{(\Gamma,\gamma)}(T^*Z_1) \to K^i_{(\Gamma,\gamma)}(T^*Z_2)$$

for a Γ-map h of the proper Γ-manifold Z_1 to the proper Γ-manifold Z_2.

Thus as in §2 we can define the geometric group also in this twisted situation, we denote it by $K^*_\gamma(X, G)$ in general, and $= K^*_\gamma(\cdot, \Gamma)$ in our special case.

Let then $C^*_r(\Gamma,\gamma)$ be the reduced C^*-algebra of the pair (Γ,γ), i.e. the C^*-algebra generated in $\ell^2(\Gamma)$ by the projective regular representation λ of Γ:

$$(\lambda(g)\xi)(g') = \gamma(g, g^{-1}g')\,\xi(g^{-1}g') .$$

As in §2 we get a map μ from $K^*_\gamma(\mathrm{pt}, \Gamma)$ to $K_*(C^*_r(\Gamma,\gamma))$, where $\mu(Z,\xi)$ is the analytical index of the K-cocycle $(Z,\xi) \in V^*_{(\Gamma,\gamma)}(T^*Z)$. The only part of the construction which is modified by the presence of γ is that of the C^*-module over $C^*_r(\Gamma,\gamma)$ attached to a (Γ,γ)-bundle E on the proper Γ-manifold Z. More precisely, one starts with the space $C_c(Z, E \otimes \Omega^{1/2})$ of compactly supported continuous $\frac{1}{2}$-density sections of E and, after choosing a Γ-invariant metric on E, one defines:

$$\langle \xi, \eta \rangle (g) = \int_X \langle \xi_x, (\eta_{xg})\, g^{-1} \rangle \quad \text{for each } g \in \Gamma ,$$

which gives a $C_c(\Gamma)$-valued sesquilinear form on $C_c(Z, E\otimes\Omega^{1/2})$. One checks that for any $\xi\in C_c(Z, E\otimes\Omega^{1/2})$, $\langle\xi,\xi\rangle$ is a *positive* element of $C_r^*(\Gamma)$, since for any $\eta\in\ell^2(\Gamma)$ one has:

$$\begin{aligned}\langle\eta,\lambda(\langle\xi,\xi\rangle)\eta\rangle &= \sum\overline{\eta}(g)\,\langle\xi,\xi\rangle\,(h)\,(\lambda(h)\,\eta)\,(g)\\ &= \sum\gamma(h,h^{-1}g)\overline{\eta}(g)\,\eta(h^{-1}g)\int_X\langle\xi_x,(\xi_x h)\,h^{-1}\rangle\\ &= \sum\overline{\eta}(g)\,\eta(h^{-1}g)\int_X\langle(\xi_{xg^{-1}})\,g,(\xi_{xg^{-1}h})\,h^{-1}g\rangle\geq 0\,.\end{aligned}$$

Then, by completion with respect to the norm $\|\langle\xi,\xi\rangle\|^{1/2}$, one gets a C^*-module over $C_r^*(\Gamma,\gamma)$, which we denote by $L^2(Z,E)$. The right action is given by:

$$(\xi f)\,(x)=\sum_\Gamma(\xi_{xg^{-1}})gf(g)\quad\text{for each }\xi\in C_c(Z,E\otimes\Omega^{1/2}),\ f\in C_c(\Gamma)\,.$$

Next, we can choose a Γ-invariant Riemannian metric on Z, represent every class in $K^0_{(\Gamma,\gamma)}(T^*Z)$ by a pair E_0, E_1 of (Γ,γ)-hermitian bundles on Z and a symbol σ which is an isomorphism of the pull back of E_0 to S^*Z to that of E_1, and is independent of ξ, $\pi(\xi)=z$, outside a Γ-compact subset of Z. Letting P_σ be the corresponding order 0 pseudo-differential operator, one gets a Kasparov $\big(\mathbf{C}, C_r^*(\Gamma,\gamma)\big)$-bimodule: the triple $(L^2(Z,E_0),\ L^2(Z,E_1),P_\sigma)$ which gives an element of $K_0(C_r^*(\Gamma,\gamma))$. It is important to give another description of the map $\mu\colon K^0_{(\Gamma,\gamma)}(T^*Z)\to K_0(C_r^*(\Gamma,\gamma))$, using Kasparov products.

PROPOSITION 1. a) *Let X be a proper Γ-manifold, then $K^i_{(\Gamma,\gamma)}(X)$ is canonically isomorphic to $K_i(C_0(X)\rtimes_\gamma\Gamma)$, where $C_0(X)\rtimes_\gamma\Gamma$ is the twisted crossed product of $C_0(X)$ by Γ.*

b) (Compare [19]). *For any C^*-algebras A, B on which Γ acts by automorphisms, one has a natural map from $KK_\Gamma(A,B)$ to $KK(A\rtimes_\gamma\Gamma, B\rtimes_\gamma\Gamma)$.*

Proof. a) One can consider $A=C_0(X)\rtimes_\gamma\Gamma$ as the C^*-algebra of the groupoid $X\rtimes\Gamma=G$ with units $G^{(0)}=X$, source and range maps $s(x,g)=xg$, $r(x,g)=x$ and composition $(x,g)\cdot(x',g')=(x,gg')$ with the 2-cocycle $\gamma\circ\pi$ where π is the natural homomorphism $G\to\Gamma$: $\pi(x,g)=g$.

Thus A is the completion of this convolution algebra $C_c(G)$:

$$(f_1*f_2)\,(x,g)=\sum_\Gamma f_1(x,h)f_2(xh,h^{-1}g)\,\gamma(h,h^{-1}g)$$

$$f^*(x,g)=\overline{f}(xg,g^{-1})$$

with the norm $\|f\| = \operatorname{Sup} \|\pi_x(f)\|$, where for each $x \in X$ the representation π_x of $C_c(G)$ in $\ell^2(\Gamma)$ is given by:

$$(\pi_x(f)\,\xi)\,(g) = \sum_{\Gamma} f(xg^{-1}, h)\,\xi(h^{-1}\,g)\,\gamma(h, h^{-1}\,g) \quad \text{for each } \xi \in \ell^2(\Gamma)\,.$$

Now, given a (Γ, γ)-vector bundle E on X, one can endow E with a Γ-invariant hermitian metric and define a C^*-module $\mathcal{E}$ over $A = C_0(X) \rtimes_\gamma \Gamma$ as follows. For any $\xi, \eta \in C_c(X, E)$ let $\langle \xi, \eta \rangle \in C_c(X \rtimes \Gamma)$ be given by $\langle \xi, \eta \rangle(x, g) = \langle \xi_x\, g, \eta_{xg} \rangle$; then $\langle \xi, \xi \rangle$ is a positive element of $A = C_0(X) \rtimes_\gamma \Gamma$, since for any $\eta \in \ell^2(\Gamma)$ and $x \in X$ one has:

$$\langle \eta, \pi_x(\langle \xi, \xi \rangle)\,\eta \rangle =$$
$$\sum \sum \langle \xi_{xg^{-1}}\,h, \xi_{xg^{-1}}\,h \rangle\,\eta(h^{-1}\,g)\,\overline{\eta}(g)\,\gamma(h, h^{-1}\,g) = \langle \alpha, \alpha \rangle \geq 0\,,$$

where $\alpha = \sum (\xi_{xg^{-1}})g\,\eta(g) \in E_x$.

Let $\mathcal{E}$ be the completion of $C_c(X, E)$ with the norm $\|\xi\| = \|\langle \xi, \xi \rangle\|$; then $\mathcal{E}$ is a C^*-module over A, with:

$$(\xi f)\,(x) = \sum f(xg^{-1}, g)\,\xi(xg')g \quad \text{for every } f \in C_c(X \rtimes \Gamma),\ \xi \in C_0(X, E)\,.$$

(One easily checks that $\langle \xi, \eta f \rangle = \langle \xi, \eta \rangle * f$ and that this right action of $C_c(X \rtimes \Gamma)$ extends to an action of A.)

The equality $(\eta\langle \eta, \xi \rangle)\,(x) = \sum \langle (\eta_{xg^{-1}})\,g, \xi_x \rangle (\eta_{xg^{-1}})\,g$ shows that any endomorphism σ of the vector bundle E which commutes with Γ and has Γ-compact support defines an A-compact endomorphism of $\mathcal{E}$ by the equality: $(T\xi)\,(x) = \sigma(x)\,\xi(x)$ for every $x \in X$. Thus, to any triple $(E_0, E_1, \sigma) \in V^0_{(\Gamma,\gamma)}(X)$ corresponds an element of $KK(\mathbf{C}, A)$, $A = C_0(X) \rtimes_\gamma \Gamma$, which obviously depends only upon the class of the triple in $K^0_{(\Gamma,\gamma)}(X)$. Let us prove that this map is an isomorphism assuming that Γ *is torsion free*. We may then assume that X is Γ-compact. We claim first that $A = C_0(X) \rtimes_\gamma \Gamma$ is Morita equivalent to a C^*-algebra with unit. Indeed, with $V = X/\Gamma$, A is the C^*-algebra of the continuous field of elementary C^*-algebras $A_t = C_0(\pi^{-1}(t)) \rtimes_\gamma \Gamma$, where $\pi \colon X \to X/\Gamma = V$ is the projection. By a simple computation, one gets that the Dixmier-Douady obstruction $\delta(A) \in H^3(V, \mathbf{Z})$ is given by $\delta(A) = \phi^*(\partial\gamma)$ where $\phi \colon V \to B\Gamma$ is the classifying map, and $\partial\gamma \in H^3(B\Gamma, \mathbf{Z})$ is the boundary of $\gamma \in H^2(B\Gamma, S^1) = H^2(\Gamma, S^1)$ in the exact sequence:

$$H^2(\Gamma, \mathbf{Z}) \to H^2(\Gamma, \mathbf{R}) \to H^2(\Gamma, S^1) \xrightarrow{\partial} H^3(\Gamma, \mathbf{Z}) \to H^3(\Gamma, \mathbf{R}) \to \dots$$

In particular $\delta(A)$ is a torsion element in $H^3(V, \mathbf{Z})$ so that there exists a bundle of matrix algebras over V with the same Dixmier-Douady obstruction and A is Morita equivalent to a unital C^*-algebra. It follows then that $K_0(A)$

is obtained from C^*-modules $\mathcal{E}$ over A with the property $\mathrm{id}_{\mathcal{E}} \in \mathrm{End}^0_A(\mathcal{E})$, i.e. all endomorphisms of $\mathcal{E}$ are A-compact. Finally, the above construction sets up a surjective map from (Γ,γ)-vector bundles on X to C^*-modules over A with the above property. Given $\mathcal{E}$, the fiber E_x of the corresponding vector bundle is:

$$E_x = \mathcal{E} \mathbin{\widehat{\otimes}_A} \ell^2(\Gamma)$$

where $A = C_0(X) \rtimes_\gamma \Gamma$ acts in $\ell^2(\Gamma)$ by the representation π_x. Since $\pi_x(A) \subset$ Compacts, one gets that E_x is a finite dimensional Hilbert space.

b) The proof is the same as in [19], one defines for any Γ-equivariant C^*-module $\mathcal{E}$ over B the crossed product $\mathcal{E} \rtimes_\gamma \Gamma$ twisted by the 2-cocycle γ. □

We can now state:

THEOREM 2. *For any element x of $K^0_{(\Gamma,\gamma)}(T^*Z) = K_0(A)$ (where $A = C_0(T^*Z) \rtimes_\gamma \Gamma$, and Z a proper Γ-manifold), one has:*

$$\mu(x) = x \otimes j_{(\Gamma,\gamma)}(D)\,,$$

where $D \in KK_\Gamma(C_0(T^*Z), \mathbf{C})$ is the class of the Dirac operator.

Note that $x \in KK(\mathbf{C}, C_0(T^*Z) \rtimes_\gamma \Gamma)$ and that

$$j_{(\Gamma,\gamma)}(D) \in KK\big(C_0(T^*Z)) \rtimes_\gamma \Gamma, C^*_r(\Gamma,\gamma)\big)\,,$$

so that the above equality is meaningful. The proof is straightforward.

To show how to use this theorem, we shall combine it with the recent result of G. G. Kasparov ([19]) to compute $K_i(C^*_r(\Gamma,\gamma))$ in the following example: we let $\Gamma = \pi_1(M)$ be the fundamental group of a Riemann surface M with genus > 1. From the exact sequence $0 \to H^2(\Gamma,\mathbf{Z}) \to H^2(\Gamma,\mathbf{R}) \to H^2(\Gamma,S^1) \to 0$ one gets $H^2(\Gamma,S^1) = \mathbf{R}/\mathbf{Z}$, so that there are many non trivial cocycles in this example. The geometric group $K^i_\gamma(\mathrm{pt},\Gamma)$ is easily determined: since the universal cover $\widetilde{M}$ of M (the Poincaré disc) is a final object in the category of proper Γ-manifolds, and homotopy classes of Γ-maps, it is enough to compute $K^i_{(\Gamma,\gamma)}(T^*\widetilde{M})$. Since $\widetilde{M}$ has a Γ-invariant Spin^c-structure, the Thom isomorphism hence gives: $K^i_\gamma(\mathrm{pt},\Gamma) = K^i_{(\Gamma,\gamma)}(\widetilde{M})$. By Proposition 1, one has $K^i_{(\Gamma,\gamma)}(\widetilde{M}) = K_i(C_0(\widetilde{M}) \rtimes_\gamma \Gamma)$ and the latter C^*-algebra is Morita equivalent to $C(M)$ (see the proof of a) in Proposition 1). Thus we get: $K^0_\gamma(\mathrm{pt},\Gamma) = \mathbf{Z}^2$, $K^1_\gamma(\mathrm{pt},\Gamma) = \mathbf{Z}^{2g}$.

THEOREM 3. *Let Γ be the fundamental group of a Riemann surface of genus >1, and $\gamma\in H^2(\Gamma,S^1)$, then the map $\mu\colon K^*_\gamma(\mathrm{pt},\Gamma)\to K_*(C^*_r(\Gamma,\gamma))$ is an isomorphism.*

Proof. Let $D\in KK_G(C_0(U),\mathbf{C})$ be the $G=PSL(2,\mathbf{R})$ equivariant Dirac operator on the Poincaré disc $U=G/G_c$ (cf. [19]). Identify $\widetilde{M}$ with U and Γ with a subgroup of G. Then by Proposition 1 b) and Theorem 2 it is enough to show that the restriction of D to an element of $KK_\Gamma(C_0(U),\mathbf{C})$ is an invertible element. This follows from [19] which shows that D is an invertible element of $KK_G(C_0(U),\mathbf{C})$, and from the multiplicative property of the restriction to subgroups.

We shall now show how to prove that the C^*-algebras $C^*_r(\Gamma,\gamma)$ are pairwise non-isomorphic when γ varies in $H^2(\Gamma,S^1)$. In fact we shall compute in full generality the composition $\zeta\circ\mu$ of the canonical trace ζ on $C^*_r(\Gamma,\gamma)$ (viewed as a map from K_0 to $\mathbf{C}$) with the above map $\mu\colon K^0_\gamma(\mathrm{pt},\Gamma)\to K_0(C^*_r(\Gamma,\gamma))$.

The computation is a generalization of the index theorem for covering spaces of Atiyah ([3]).

LEMMA 4. *Let Z be a proper Γ-manifold and E a (Γ,γ) vector bundle on Z. There exists a Γ-invariant connection ∇ on E.*

Proof. For any (Γ,γ)-vector bundle F on Z and section $\xi\in C^\infty_c(Z,F)$ let, for $g\in\Gamma$, $g\xi\in C^\infty_c(Z,F)$ be given by: $(g\xi)(x)=(\xi(xg))\,g^{-1}\in F_x$ for every $x\in Z$.

In this way one gets a natural γ-action of Γ on both $C^\infty_c(Z,E)$ and $C^\infty_c(Z,E\otimes T^*Z)$, and one looks for a connection

$$\nabla\colon C^\infty_c(Z,E)\to C^\infty_c(Z,E\otimes T^*Z)$$

such that $\nabla(g\xi)=g(\nabla\xi)$ for every ξ. Let $f\in C^\infty(Z)$, $0\le f\le 1$, be such that $\sum_\Gamma f(xg)=1$ for every $x\in Z$ and ∇_0 be a connection on E. Put $\nabla=\sum_\Gamma g^{-1}(f\nabla_0)g$. By construction ∇ is Γ-invariant, moreover each $g^{-1}\nabla_0\, g$ is a connection on E thus ∇ is a connection on E. $\square$

Proof of Theorem 3, continued. Assuming now that Z is Γ-compact, let for a Γ-invariant connection ∇ on E, ω_∇ be the canonical differential form on Z which represents locally the Chern character $\mathrm{ch}(E)$. By construction ω_∇ is Γ-invariant and hence determines a cohomology class in Z/Γ. One checks as usual that this class does not depend upon the choice of ∇ and we shall denote it by $[E] \in H^*(Z/\Gamma, \mathbf{R})$. This construction easily extends to give a map ch from $K^0_{(\Gamma,\gamma)}(Z)$ to $H^*(Z/\Gamma, \mathbf{R})$ for any proper Γ-manifold Z. However, in the presence of the 2-cocycle γ the range of this map is *no longer necessarily contained* in $H^*(Z/\Gamma, \mathbf{Q})$.

To be more precise, let us make a few simplifying assumptions and compute exactly the range of this Chern character:

$$\mathrm{ch}\colon K^0_{(\Gamma,\gamma)}(Z) \to H^*(Z/\Gamma, \mathbf{R}).$$

Thus let us assume that Γ is torsion free and that the image of $\gamma \in H^2(\Gamma, S^1)$ in $H^3(\Gamma, \mathbf{Z})$ under the connecting map of the long exact sequence:

$$\ldots \to H^2(\Gamma, \mathbf{Z}) \to H^2(\Gamma, \mathbf{R}) \to H^2(\Gamma, S^1) \to H^3(\Gamma, \mathbf{Z}) \to \ldots$$

is equal to 0 (it is always a torsion element).

Let then $\rho \in H^2(\Gamma, \mathbf{R})$ be such that $e(\rho) = \gamma$ where $e\colon \mathbf{R} \to S^1$ is given by $e(s) = \exp(2\pi i s)$, for each $s \in \mathbf{R}$.

LEMMA 5. a) *Let* $\rho \in Z^2(\Gamma, \mathbf{R})$ *and* Z *be a proper* Γ*-manifold, then there exists a smooth function* $c \in C^\infty(Z \rtimes \Gamma)$ *such that:*

$$c(x, g_1) + c(xg_1, g_2) = c(x, g_1 g_2) - \rho(g_1, g_2)$$

for every $x \in Z$, $g_1, g_2 \in \Gamma$.

b) *If* $\gamma = e(\rho)$ *there exists an isomorphism* $r\colon K^0_\Gamma(Z) \to K^0_{(\Gamma,\gamma)}(Z)$ *making the following diagram commutative:*

$$\begin{array}{ccc} K^0_\Gamma(Z) & \xrightarrow{\ r\ } & K^0_{(\Gamma,\gamma)}(Z) \\ \downarrow{\scriptstyle \mathrm{ch}} & & \downarrow{\scriptstyle \mathrm{ch}} \\ H^*(Z/\Gamma) & \xrightarrow{\ m\ } & H^*(Z/\Gamma), \end{array}$$

where m *is multiplication by the cohomology class* $\exp(\phi^*\rho)$ *and where* $\phi\colon Z/\Gamma \to B\Gamma$ *is the classifying map.*

Proof. a) Let $M = Z/\Gamma$, $\pi\colon Z \to M$ the projection. Since Z is a locally trivial Γ-principal bundle, it is easy to construct c on the open set $\pi^{-1}(U)$ for U small enough. Then one combines such c_U by a smooth partition of unity on M:

$$c(x, g) = \sum \phi_U(\pi(x))\, c_U(x, g).$$

b) Let $c \in C^\infty(Z \rtimes \Gamma)$ be as in a) and let us endow the trivial line bundle on Z (with total space $Z \times \mathbf{C}$) with a structure of (Γ, γ)-bundle. We take:

$$(x,\lambda)g = (xg, e(c(x,g))\lambda).$$

(One has $((x,\lambda)g_1)g_2 = (xg_1g_2, e(c(x,g_1)+c(xg_1,g_2))\lambda) = \gamma^{-1}(g_1,g_2)(x\lambda)(g_1g_2)$.)

Let L be the (Γ,γ)-line bundle on Z thus obtained. It is obvious that tensoring by L gives an isomorphism of $V^0_{(\Gamma)}(Z)$ with $V^0_{(\Gamma,\gamma)}Z$ and hence of $K^0_\Gamma(Z)$ with $K^0_{(\Gamma,\gamma)}(Z)$. □

End of proof of Theorem 3. To conclude, it is enough to compute $\mathrm{ch}(L)$. Let $\xi \in C^\infty(Z,L)$ be the section $\xi(x) = 1$ for every $x \in Z$. Let ∇ be a Γ-invariant connection on L, one has $\mathrm{ch}(L) = \exp(\omega)$ where $\omega \in H^2(Z/\Gamma, \mathbf{R})$ corresponds to the Γ-invariant 2-form $\theta = \frac{1}{2\pi i} d(\nabla\xi/\xi)$ on Z. Let $\alpha = \frac{1}{2\pi i}\nabla\xi/\xi$, then α is a 1-form on Z, and let us compute for any $g \in \Gamma$ the difference $\alpha - \phi^*\alpha$ where $\phi(x) = xg$ for every $x \in Z$. Since ∇ is Γ-invariant, one has $\phi^*\alpha = \frac{1}{2\pi i}\nabla g(\xi)/g(\xi)$, and as $g(\xi)(x) = e(c(xg,g^{-1}))\,\xi(x)$ one gets $\phi^*\alpha - \alpha = d\psi_g$, where $\psi_g(x) = c(xg, g^{-1})$ for every $x \in Z$. One has $\psi_{g_1g_2} - g_1\psi_{g_2} - \psi_{g_1} = \rho(g_2^{-1}, g_1^{-1})$. This shows that the class of θ in $H^2(Z/\Gamma,\mathbf{R})$ is the pull back of the class of $-\rho$ in $H^2(B\Gamma,\mathbf{R})$, by the classifying map: $Z/\Gamma \to B\Gamma$. □

Using this map $\mathrm{ch}\colon K^*_{(\Gamma,\gamma)}(Z) \to H^*(Z/\Gamma,\mathbf{R})$ we get, by the same five steps as in §6, a map

$$K^*_\gamma(\mathrm{pt},\Gamma) \xrightarrow{\mathrm{ch}} H_*(B\Gamma,\mathbf{R}).$$

Again as in §6, let ϵ be the map from $B\Gamma$ to a point, and tr_Γ be the canonical trace on $C^*_r(\Gamma,\gamma)$.

THEOREM 6. *For any discrete group Γ and 2-cocycle γ the following diagram is commutative:*

$$\begin{array}{ccc} K^0_\gamma(\mathrm{pt},\Gamma) & \xrightarrow{\mu} & K_0(C^*_r(\Gamma,\gamma)) \\ \big\downarrow{\scriptstyle \mathrm{ch}} & & \big\downarrow{\scriptstyle \mathrm{tr}_\Gamma} \\ H_*(B\Gamma,\mathbf{R}) & \xrightarrow{\epsilon^*} & \mathbf{C}. \end{array}$$

The proof is a simple adaptation of the heat equation method to compute the Γ-index of the (Γ,γ)-Dirac operator on a Γ-manifold Z.

COROLLARY 7. *If* $\gamma = e(\rho)$, *for some* $\rho \in H^2(\Gamma, \mathbf{R})$, *then the subgroup of* $\mathbf{R}$, $\Delta = \mathrm{tr}_\Gamma(K_0(C_r^*(\Gamma,\gamma)))$ *contains the group:*

$$\langle \mathrm{ch}\, K_*(B\Gamma), \exp(\rho)\rangle .$$

This follows from Theorem 6 and Lemma 5 b).

Moreover, when the map μ is an isomorphism, one can conclude that $\Delta = \langle \mathrm{ch}\, K_*(B\Gamma), \exp(\rho)\rangle$. Thus using Theorem 3 we get:

COROLLARY 8. *Let* Γ *be the fundamental group of a compact Riemann surface of positive genus,* $\gamma \in H^2(\Gamma, S^1)$ *be a 2-cocycle and* $\theta \in \mathbf{R}/\mathbf{Z}$ *the class of* γ *in* $H^2(\Gamma,\mathbf{R})/H^2(\Gamma,\mathbf{Z}) = \mathbf{R}/\mathbf{Z}$. *Then the image of* $K_0(C_r^*(\Gamma,\gamma))$ *by the canonical trace* $\zeta = \mathrm{Tr}_\Gamma$ *is equal to the subgroup* $\mathbf{Z} + \theta\mathbf{Z} \subset \mathbf{R}$.

Since, for $g > 1$, the trace tr_Γ is the unique normalized trace on $C_r^*(\Gamma,\gamma)$ (for any value of γ), one gets that the corresponding C^*-algebras are isomorphic only when the Γ's are the same (using K_1) and when the γ's are equal or opposite (in $H^2(\Gamma, S^1)$).

9. FOLIATIONS

Let V be a C^∞-manifold, and let F be a C^∞-foliation of V. Thus F is a C^∞-integrable sub-vector bundle of TV. As in [33] let G be the holonomy groupoid (graph) of (V,F). The manifold V is assumed to be Hausdorff and second countable. G, however, is a C^∞-manifold which might not be Hausdorff. A point in G is an equivalence class of C^∞-paths

$$\gamma\colon [0,1] \to V$$

such that $\gamma(t)$ remains within one leaf of the foliation for all $t \in [0,1]$. Set $s(\gamma) = \gamma(0)$, $r(\gamma) = \gamma(1)$. The equivalence relation on the γ preserves $s(\gamma)$ and $r(\gamma)$ so G comes equipped with two maps $G \underset{r}{\overset{s}{\rightrightarrows}} V$.

Let Z be a possibly non-Hausdorff C^∞-manifold. Assume given a C^∞-map $\rho\colon Z \to V$, set

$$Z \circ G = \{(z,\gamma) \in Z \times G \mid \rho(z) = s(\gamma)\} .$$

A C^∞ *right action of* G *on* Z is a C^∞-map

$$Z \circ G \to Z$$

denoted by

$$(z, \gamma) \to z\gamma$$

such that

$$\rho(z\gamma) = r(\gamma)\,, \quad (z\gamma)\,\gamma' = z(\gamma\gamma')\,, \quad (zl_p) = z\,,$$

where l_p denotes the constant path at $p \in V$.

An action of G on Z is *proper* if:

(i) the map $Z \circ G \to Z \times Z$ given by $(z, \gamma) \mapsto (z, z\gamma)$ is proper (i.e. the inverse image of a compact set is compact);

(ii) the quotient space Z/Γ is Hausdorff. Here Z/Γ is the set of equivalence classes of $z \in Z$ where $z \sim z'$ if, for some $\gamma \in G$, $z\gamma = z'$.

Specializing to $Z = V$, the groupoid G acts on V by $\rho(p) = p$ and

$$p\gamma = \gamma(1)$$

($p \in V$, $\gamma \in G$, $p = \gamma(0)$). For many examples this action of G on V is not proper. Set $\nu_p = T_pV/F_p$, so that ν is the normal bundle of the foliation. ν is a G-vector bundle since the derivative of holonomy gives a linear map

$$\nu_p \mapsto \nu_{p\gamma}\,.$$

This is, of course, just the well-known fact that ν is flat along the leaves of the foliation.

More generally, if Z is a G-manifold, then the orbits of the G-action foliate Z. Denote the normal bundle of this foliation by $\widetilde{\nu}$. Then $\widetilde{\nu}$ is a G-vector bundle on Z.

If Z is a proper G-manifold, a G-vector bundle on Z with G-compact support is a triple (E_0, E_1, σ) where E_0, E_1 are G-vector bundles on Z and $\sigma\colon E_0 \to E_1$ is a morphism of G-vector bundles with Support(σ) G-compact. As in §2 above one then defines $V^i_G(Z)$ and $K^i_G(Z)$, $i = 0, 1$. These are defined and used *only* for proper G-manifolds.

DEFINITION 1. A *K-cocycle for* (V, F) is a pair (Z, ξ) such that

(1) Z is a proper G-manifold,

(2) $\xi \in V^*_G[(\widetilde{\nu})^* \oplus \rho^*\nu^*]$, where $\rho\colon Z \to V$ is given by the action of G on Z.

In [12] and [14] a canonical C^*-algebra $C^*(V, F)$ is constructed. This C^*-algebra can heuristically be thought of (up to Morita equivalence) as the algebra of continuous functions on the "space of leaves" of the foliation. Thus $K_*C^*(V,F)$ can be viewed as the K-theory of the "space of leaves" of the foliation.

To define the geometric K-theory $K^*(V,F)$ we proceed quite analogously to §2 above.

THEOREM 2. *Let* (Z,ξ) *be a cocycle for* (V,F). *Then* (Z,ξ) *determines an element in* $K_*C^*(V,F)$.

Proof. If $\rho\colon Z \to V$ is a submersion then ξ gives rise to the symbol of a G-equivariant family of elliptic operators D, parametrized by the points of V. The K-theory index of this family D is the desired element of $K_*C^*(V,F)$.

If $\rho\colon Z \to V$ is not a submersion, then as in the proof of Theorem 1 of §2 one reduces to the submersion case. □

REMARK 3. With D as in the proof of the Theorem, Index$(D) \in K_*C^*(V,F)$ will be denoted $\mu(Z,\xi)$. For $\xi \in V_G^i[(\widetilde{\nu})^* \oplus \rho^*\nu^*]$, $\mu(Z,\xi) \in K_i\,C^*(V,F)$, $i = 0, 1$.

Suppose given a commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{\ h\ } & Z_2 \\ & {\scriptstyle \rho_1}\searrow \quad \swarrow{\scriptstyle \rho_2} & \\ & V & \end{array}$$

where Z_1, Z_2 are G-manifolds with Z_1, Z_2 proper and h is a G-map. There is then a Gysin map

$$h_!\colon K_G^i[(\widetilde{\nu}_1)^* \oplus \rho_1^*\nu^*] \to K_G^i[(\widetilde{\nu}_2)^* \oplus \rho_2^*\nu^*].$$

THEOREM 4. *If* $\xi_1 \in V_G^*[(\widetilde{\nu}_1)^* \oplus \rho_1^*\nu]$ *then* $\mu(Z_1,\xi_1) = \mu(Z_2, h_!(\xi_1))$.

REMARK 5. Let $\Gamma(V,F)$ be the collection of all K-cocycles (Z,ξ) for (V,F). On $\Gamma(V,F)$ impose the equivalence relation $\sim$, where $(Z,\xi) \sim (Z',\xi')$ if and only if there exists a commutative diagram

$$\begin{array}{ccccc} Z & \xrightarrow{\ h\ } & Z'' & \xleftarrow{\ h'\ } & Z' \\ & {\scriptstyle \rho}\searrow & \downarrow{\scriptstyle \rho''} & \swarrow{\scriptstyle \rho'} & \\ & & V & & \end{array}$$

with h and h' G-maps and with $h_!(\xi) = h'_!(\xi')$.

DEFINITION 6. $K^*(V,F) = \Gamma(V,F)/\sim$. Addition in $K^*(V,F)$ is by disjoint union of K-cocycles. The natural homomorphism of abelian groups

$$K^i(V,F) \to K_i\, C^*(V,F)$$

is defined by

$$(Z,\xi) \to \mu(Z,\xi)\,.$$

CONJECTURE. *$\mu\colon K^*(V,F) \to K_* C^*(V,F)$ is an isomorphism.*

REMARK 7. Calculations of M. Pennington [25] and A.M. Torpe [32] verify the conjecture for certain foliations.

Given (V,F), let BG be the classifying space of the holonomy groupoid G. Since ν is a G-vector bundle on V, ν induces a vector bundle τ on BG. As in §3 above there is then a natural map

$$K_*^\tau(BG) \to K^*(V,F)\,.$$

PROPOSITION 8. *The natural map $K_*^\tau(BG) \to K^*(V,F)$ is rationally injective. If G is torsion free then $K_*^\tau(BG) \to K^*(V,F)$ is an isomorphism.*

REMARK 9. Examples show that for foliations with torsion holonomy, the map $K_*^\tau(BG) \to K^*(V,F)$ may fail to be an isomorphism.

THEOREM 10. *If F admits a C^∞ Euclidean structure such that the Riemannian metric for each leaf has all sectional curvatures non-positive, then*

$$\mu\colon K^*(V,F) \to K_* C^*(V,F)$$

is injective.

10. FURTHER DEVELOPMENTS

The theory outlined in §§1–8 can be developed in various directions. We very briefly mention two of them here.

Let A be a C^*-algebra. If G is a Lie group and X is a G-manifold, then using A as coefficients there is both a geometric and an analytic K-theory for (X,G). The analytic K-theory is the K-theory of the C^*-algebra $(C_0(X) \rtimes G) \otimes A$.

The geometric K-theory is obtained from K-cocycles (Z,ξ,f) where Z,f are as in §2 and $\xi = \{E_0 \xrightarrow{\sigma} E_1\}$ uses G-vector bundles E_0, E_1 on $T^*Z \oplus f^*T^*X$ such that the fibres of E_i are finitely generated projective modules over A. Denote this geometric K-theory by $K^*(X,G;A)$. The natural map

$$K^i(X,G;A) \to K_i[(C_0(X) \rtimes G) \otimes A]$$

is defined by using elliptic operators in the spirit of Miscenko-Fomenko [22]. We conjecture that this natural map is an isomorphism.

In the notation of Kasparov [18] the group denoted here by $K_*[C_0(X) \rtimes G]$ is $KK(\mathbf{C}, C_0(X) \rtimes G)$. For the K-homology group $KK(C_0(X) \rtimes G, \mathbf{C})$ there is a geometric group $K_*(X,G)$ which is the G-equivariant version of the topologically defined K-homology of [9]. Using transversally elliptic operators [2] one then obtains a natural map

$$K_*(X,G) \to KK(C_0(X) \rtimes G, \mathbf{C}).$$

We conjecture that this map is injective and that its image is dense (with respect to the natural topology) in $KK(C_0(X) \rtimes G, \mathbf{C})$.

11. Acknowledgements

We are grateful to A. Borel for communicating us

Proposition 1 (A. Borel [10]). *Let G be a Lie group with $\pi_0 G$ finite and maximal compact subgroup H. If Z is any proper G-manifold then there exists a G-map from Z to $H\backslash G$.*

In §5 above this was proved for G a connected semi-simple Lie group with finite center. By the argument of §5, Borel's result implies:

Corollary 2. *Let G be a Lie group with $\pi_0 G$ finite. For any G-manifold X there is an isomorphism of abelian groups*

$$K^i_H(X \times (\mathfrak{h}\backslash\mathfrak{g})^*) \to K^i(X,G) \qquad (i = 0,1).$$

Finally, it is a pleasure to thank the many mathematicians who have given us stimulating and enlightening comments. We thank A. Borel, J. Cohen, R. Douglas, B. Lawson, G. Mackey, R. MacPherson, J. Rosenberg, and G. Schwarz.

REFERENCES

[1] ATIYAH, M. F. Bott periodicity and the index of elliptic operators. *Quart. J. Math. Oxford Ser. (2) 19* (1968), 113–140.

[2] —— *Elliptic Operators and Compact Groups.* Lecture Notes in Math. 401. Springer-Verlag, 1974.

[3] —— Elliptic operators, discrete groups and von Neumann algebras. *Astérisque 32/33* (1976), 43–72.

[4] ATIYAH, M. F. and W. SCHMID. A geometric construction of the discrete series for semisimple Lie groups. *Invent. Math. 42* (1977), 1–62.

[5] ATIYAH, M. F. and G. B. SEGAL. The index of elliptic operators, II. *Ann. of Math. (2) 87* (1968), 531–545.

[6] ATIYAH, M. F. and I. M. SINGER. The index of elliptic operators, I. *Ann. of Math. (2) 87* (1968), 484–530; The index of elliptic operators, III. *Ann. of Math. (2) 87* (1968), 546–604.

[7] ATIYAH, M. F. and I. M. SINGER. The index of elliptic operators, IV. *Ann. of Math. (2) 93* (1971), 119–138.

[8] ATIYAH, M. F. and I. M. SINGER. The index of elliptic operators, V. *Ann. of Math. (2) 93* (1971), 139–149.

[9] BAUM, P. and R. DOUGLAS. K-homology and index theory. *Operator algebras and applications, Proc. Symposia Pure Math. 38* (1982), Part I, 117–173.

[10] BOREL, A. Letter to P. Baum, 1982.

[11] CAPPELL, S. E. On homotopy invariance of higher signatures. *Invent. Math. 33* (1976), 171–179.

[12] CONNES, A. Sur la théorie non commutative de l'intégration. *Algèbres d'opérateurs.* Lecture Notes in Math. 725, Springer-Verlag, (1979), 19–143.

[13] —— An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbf{R}$. *Adv. in Math. 39* (1981), 31–55.

[14] —— A survey of foliations and operator algebras. *Operator algebras and applications. Proc. Symposia in Pure Math. 38* (1982), Part 1, 521–628.

[15] CONNES, A. and H. MOSCOVICI. The L^2-index theorem for homogeneous spaces of Lie groups. *Ann. of Math. (2) 115* (1982), 291–330.

[16] ELLIOTT, G. A. On the K-theory of the C^*-algebra generated by a projective representation of a torsion-free, discrete abelian group. In: *Operator Algebras and Group Representations*, Vol. I (Neptun, 1980), *Monographs Stud. Math. 17*, Pitman (1985), 157–184.

[17] HELGASON, S. *Differential Geometry and Symmetric Spaces.* Academic Press, 1962.

[18] KASPAROV, G. G. The operator K-functor and extensions of C^*-algebras. *Izv. Akad. Nauk SSSR, Ser. Mat. 44* (1980), 571–636; *Math. USSR Izv. 16* (1981), 513–572.

[19] —— K-theory, group C^*-algebras, and higher signatures (Conspectus), (first distributed 1981), In: *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 1, S.C. Ferry, A. Ranicki and J. Rosenberg Editors, London Math. Soc. Lecture Notes 226, Cambridge Univ. Press (1995), 101–146.

[20] —— Index of invariant elliptic operators, K-theory and representations of Lie groups (Russian). *Dokl. Akad. Nauk SSSR 268 no 3* (1983), 533–537.

[21] MISCENKO, A. S. C^*-algebras and K-theory, *Algebraic Topology, Aarhus 1978.* Lecture Notes in Math. 763, Springer-Verlag, 1979, 262–274.

[22] MISCENKO, A. S. and A. T. FOMENKO. The index of elliptic operators over C^*-algebras. *Izv. Akad. Nauk SSSR, Ser. Mat. 43* (1979), 831–859; *Math. USSR Izv. 15* (1980), 87–112.

[23] PALAIS, R. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2) 73* (1961), 295–323.

[24] PASSMAN, D. S. *The Algebraic Structure of Group Rings.* Wiley-Interscience, 1977.

[25] PENNINGTON, M. K-theory and C^*-algebras of Lie groups and foliations. PhD Thesis, Oxford, Michaelmas term, 1983. (Part of this has appeared in [25′].)

[25′] PENNINGTON, M. and R. J. PLYMEN. The Dirac operator and the principal series for complex semisimple Lie groups. *J. Funct. Anal. 53* (1983), 269–286.

[26] PIMSNER, M. and D. VOICULESCU. Exact sequences for K-groups and Ext-group of certain crossed product C^*-algebras. *J. Operator Theory 4* (1980), 93–118.

[27] PIMSNER, M. and D. VOICULESCU. K-groups of reduced crossed products by free groups. *Increst preprint series in Math. 62* (1981).

[28] PLYMEN, R. Proof of Connes' conjecture for complex semi-simple groups. Univ. of Manchester preprint, 1982. See also M. G. Pennington and R. J. Plymen, The Dirac operator and the principal series for complex semisimple Lie groups. *J. Funct. Anal. 53* (1983), 269–286.

[29] RIEFFEL, M. A. C^*-algebras associated with irrational rotations. *Pacific J. Math. 93* (1981), 415–429.

[30] ROSENBERG, J. C^*-algebras, positive scalar curvature, and the Novikov conjecture. *Publ. Math. I.H.E.S. 58* (1983), 197–212. (Appendix to "Positive scalar curvature and the Dirac operator on complete Riemannian manifolds" by M. Gromov and H. B. Lawson, Jr. *Publ. Math. I.H.E.S. 58* (1983), 83–196.)

[31] SEGAL, G. B. Equivariant K-theory. *Publ. Math. I.H.E.S. 34* (1968), 129–151.

[32] TORPE, A. M. K-theory for the leaf space of foliations by Reeb components. *J. Funct. Anal. 61* (1985), 15–71.

[33] WINKELNKEMPER, H. E. The graph of a foliation. *Ann. Global Anal. Geom. 1* (1983), 51–75.

EDITORS' NOTE

The paper by Paul Baum and Alain Connes has been printed here as it was circulated in 1982, without change. (But we have updated the references that were preprints in 1982 and have appeared since; and a small number of typographical corrections have been made.) The subject has developed considerably over the last 18 years, as testified by the supplementary bibliography below, for which we express our debt to Alain Valette.

We would also like to make the following remarks:

(i) The conjecture on the invariance of certain foliation characteristic classes, alluded to at the end of Section 1 in the above paper, has appeared in:

BAUM, P. and A. CONNES. Leafwise homotopy equivalence and rational Pontrjagin classes. In: *Foliations (Tokyo, 1983).* Adv. Stud. Pure Math. *5*, 1–14. North-Holland, 1985.

(ii) Concerning Corollary 5 and Remark 7 of §5 above, Cédric Béguin has observed that it is necessary to assume that G is connected (and not only that $\pi_0(G)$ is finite), as is shown by the example in which G is the group $\{x \mapsto ax + b \mid a \in \mathbf{R}^*, b \in \mathbf{R}\}$ and H the subgroup $\{x \mapsto \pm x\}$. Indeed, the connectedness of H is used by Baum and Connes just after their Remark 4. This observation is repeated from

BÉGUIN, C. Autour de la conjecture des idempotents. Thèse, Université de Neuchâtel, 1999.

(iii) The conjecture on the range of the map $\mathrm{tr}\colon K_0C^*G \longrightarrow \mathbf{R}$ for a group G with torsion, stated just before Corollary 4 in §7, has been *disproved* in:

ROY, R. The trace conjecture – a counterexample. *K-Theory 17* (1997), 209–213.

Roy's example indicates that, if G has p-torsion (p a prime) then higher powers of p in denominators do appear in the range of the trace map.

(iv) Finally, we would like to mention that the paper by P. Baum and A. Connes has motivated several books still in preparation, including one by J. Roe and N. Higson, and one by A. Valette.

FURTHER REFERENCES ON THE BAUM-CONNES CONJECTURE

(compiled and classified by Alain VALETTE)

1) *K-homology*

ATIYAH, M.F. Global theory on elliptic operators. *Proc. Int. Symp. on Functional Analysis.* Univ. of Tokyo Press, Tokyo, 1969, 21–30.

—— Elliptic operators, discrete groups and von Neumann algebras. *Astérisque 32-33* (1976), 43–72.

BAUM, P. and R. G. DOUGLAS. *K*-theory and index theory. *Proc. Symp. Pure Math. 38.* Part 1 (Operator Algebras and Applications, R. Kadison ed.), Amer. Math. Soc. (1982), 117-173.

KASPAROV, G. G. Topological invariants of elliptic operators, I: *K*-homology (English translation). *Math. USSR Izvestija 9* (1975), 751–792.

2) *Chern characters*

BAUM, P., J. BLOCK and N. HIGSON. Chern character for totally disconnected groups. Preprint.

BAUM, P. and A. CONNES. Chern character for discrete groups. *A Fête of Topology.* North Holland, 1987, 163–232.

BAUM, P., N. HIGSON and R. J. PLYMEN. Equivariant homology for SL(2) of a p-adic field. *Contemp. Math. 148* (1993), 1–18.

HIGSON, N. and V. NISTOR. Cyclic homology of totally disconnected groups acting on buildings. *J. Funct. Anal. 141* (1996), 466–495.

SCHNEIDER, P. The cyclic homology of p-adic reductive groups. *J. reine angew. Math. 475* (1996), 39–54.

3) *Kasparov theory (KK-theory)*

BAAJ, S. and P. JULG. Théorie bivariante de Kasparov et opérateurs non bornés dans les C^*-modules hilbertiens. *C. R. Acad. Sci. Paris, Sér. I Math. 296* (1983), 875–878.

CUNTZ, J. and G. SKANDALIS. Mapping cones and exact sequences in *KK*-theory. *J. Operator Theory 15* (1986), 163–180.

KASPAROV, G. G. The operator *K*-functor and extensions of C^*-algebras (English translation). *Math. USSR Izvestyia 16* (1981), 513–572.

—— An index for invariant elliptic operators, *K*-theory, and representations of Lie groups (English translation). *Soviet Math. Dokl. 27* (1983), 105–109.

—— Operator *K*-theory and its applications: elliptic operators, group representations, higher signatures, C^*-extensions. *Proc. ICM*, vol. 2, Warsaw (1983), 987–1000.

KNUDSEN JENSEN, M. and K. THOMSEN. *Elements of KK-Theory.* Birkhäuser, 1991.

KUCEROVSKI, D. Kasparov products in KK-theory and unbounded operators with applications to index theory. PhD thesis, Magdalen College, Oxford, 1994.

PIMSNER, M. KK-groups of crossed products by groups acting on trees. *Invent. Math. 86* (1986), 603–634.

4) *E-theory*

CONNES, A. and N. HIGSON. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris, Sér. I Math. 311* (1990), 101–106.

GUENTNER, E., N. HIGSON and J. TROUT. Equivariant theory for C^*-algebras. Preprint, 1997.

HIGSON, N., G. G. KASPAROV and J. TROUT. A Bott periodicity theorem for infinite-dimensional Euclidian space. *Adv. Math. 135* (1998), 1–40.

5) *K-amenability*

CUNTZ, J. K-theoretic amenability for discrete groups. *J. reine angew. Math. 344* (1983), 180–195.

FOX, J. and P. HASKELL. K-amenability for $SU(n,1)$. *J. Funct. Anal. 117* (1993), 279–307.

JULG, P. and A. VALETTE. K-theoretic amenability for $SL_2(\mathbf{Q}_p)$ and the action on the associated tree. *J. Funct. Anal. 58* (1984), 194–215

SKANDALIS, G. Une notion de nucléarité en K-théorie. *K-Theory 1* (1988), 549–573.

6) *Connes-Kasparov conjecture (connected Lie groups)*

CONNES, A. An analogue of the Thom isomorphism for crossed products of a C^*-algebra by an action of $\mathbf{R}$. *Adv. Math. 39* (1981), 31–55.

PENINGTON, M. and R. J. PLYMEN The Dirac operator and the principal series for complex semi-simple Lie groups. *J. Funct. Anal. 53* (1983), 269–286.

VALETTE, A. K-theory for reduced C^*-algebra of a semisimple Lie group with rank 1 and finite centre. *Quart. J. Math. Oxford Ser. (2) 35* (1984), 341–359.

—— Dirac induction for semi-simple Lie groups having one conjugacy class of Cartan subgroups. *Proc. Conf. on Operator algebras, ergodic theory and topology*, Busteni (Roumanie) 1983. Springer Lecture Notes in Math. 1132 (1985), 526–555.

WASSERMANN, A. Une démonstration de la conjecture de Connes-Kasparov pour les groupes de Lie linéaires connexes réductifs. *C. R. Acad. Sci. Paris, Sér. I Math. 304* (1987), 559–562.

7) *Baum-Connes conjecture – surjectivity of μ*

BAUM, P. and A. CONNES. Geometric K-theory for Lie groups and foliations. Preprint, 1982. *L'Enseignement Math. (2) 46* (2000), 3–42.

—— *K-theory for Discrete Groups in Operator Algebras and Applications.* D. Evans and M. Takesaki eds. Cambridge Univ. Press (1988), 1–20.

BAUM, P., A. CONNES and N. HIGSON. Classifying space for proper actions and K-theory of group C^*-algebras. In *C^*-Algebras: 1943-1993, a Fifty Year Celebration*, R. S. Doran ed. *Contemp. Math. 167* (1994), 240–291.

BAUM, P., N. HIGSON and R. J. PLYMEN. A proof of the Baum-Connes conjecture for p-adic $\mathrm{GL}(n)$. *C. R. Acad. Sci. Paris, Sér. I Math. 325* (1997), 171–176.

BÉGUIN, C., H. BETTAIEB and A. VALETTE. K-theory for C^*-algebras of one-relator groups. *K-Theory 16* (1999), 277–298.

BLOCK, J. Some remarks concerning the Baum-Connes conjecture. *Comm. Pure Appl. Math. 50* (1997), 813–820.

CONNES, A. *Noncommutative Geometry.* Academic Press, 1994.

GROMOV, M. Positive curvatures, macroscopic dimension, spectral gaps, and higher signatures. *Functional Analysis on the Eve of the 21st Century*, Vol. II (New Brunswick, NJ, 1993), *Progr. Math. 132* (1996), 1–213.

HIGSON, N. and G. G. KASPAROV. Operator K-theory for groups which act properly and isometrically on Hilbert space. *Electron. Res. Announc. Amer. Math. Soc. 3* (1997), 131–142.

JULG, P. Remarks on the Baum-Connes conjecture and Kazhdan's property (T). *Fields Inst. Commun. 13* (1997), 145–155.

—— Travaux de N. Higson et G. Kasparov sur la conjecture de Baum-Connes. *Séminaire Bourbaki*, Exposé 841 (1998).

JULG, P. and G. G. KASPAROV. Operator K-theory for the group $\mathrm{SU}(n, 1)$. *J. reine angew. Math. 463* (1995), 99–152.

KASPAROV, G. G. Lorentz groups: K-theory for unitary representation and crossed products. (English translation). *Soviet Math. Dokl. 29* (1984), 256–260.

KESWANI, N. Homotopy invariance of relative eta-invariants and C^*-algebra K-theory. *Electron. Res. Announc. Amer. Math. Soc. 4* (1998), 18–26.

LAFFORGUE, V. Une démonstration de la conjecture de Baum-Connes pour les groupes réductifs sur un corps p-adique et pour certains groupes discrets possédant la propriété (T). *C. R. Acad. Sci. Paris, Sér. I Math. 327* (1998), 439–444.

—— Compléments à la démonstration de la conjecture de Baum-Connes pour certains groupes possédant la propriété (T). *C. R. Acad. Sci. Paris, Sér. I Math. 328* (1999), 203–208.

OYONO-OYONO, H. La conjecture de Baum-Connes pour les groupes agissant sur les arbres. *C. R. Acad. Sci. Paris, Sér. I Math. 326* (1998), 799–804.

—— Baum-Connes conjecture and extensions. Preprint, September 1999.

ROSENBERG, J. Group C^*-algebras and topological invariants. *Monographs Stud. Math. 18*, Pitman (1984), 95–115.

SKANDALIS, G. Progrès récents sur la conjecture de Baum-Connes. Contribution de Vincent Lafforgue. Sém. Bourbaki (1999), Exposé 869.

TU, J.-L. The Baum-Connes conjecture and discrete group actions on trees. *K-Theory 17* (1999), 303–318.

VALETTE, A. The conjecture of idempotents: a survey of the C^*-algebraic approach. *Bull. Soc. Math. Belg. Sér. A 41* (1989), 485–521.

—— Introduction to the Baum-Connes conjecture. Preprint, January 2000, http://www.math.ethz.ch/~indira/Valette.dvi.

8) *Injectivity of μ – Novikov conjecture*

BETTAIEB, H. and A. VALETTE. Sur le groupe K_1 des C^*-algèbres de groupes discrets. *C. R. Acad. Sci. Paris, Sér. I Math. 322* (1996), 925–928.

CONNES, A. and H. MOSCOVICI. Cyclic cohomology, the Novikov conjecture and hyperbolic groups. *Topology 29* (1990), 345–388.

ELLIOTT, G. and T. NATSUME. A Bott periodicity map for crossed products of C^*-algebras by discrete groups. *K-Theory 1* (1987), 423–435.

FERRY, S., A. RANICKI and J. ROSENBERG. A history and survey of the Novikov conjecture. *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 1, S. Ferry, A. Ranicki and J. Rosenberg eds. London Math. Soc. Lecture Note Ser. 226. Cambridge Univ. Press, 1995, 7–66.

HIGSON, N. Bivariant K-theory and the Novikov conjecture. Preprint, summer 1999.

HIGSON, N. and J. ROE. On the coarse Baum-Connes conjecture. *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 2. London Math. Soc. Lecture Note Ser. 227 (1995), 227–254.

—— Amenable group actions and the Novikov conjecture. Preprint, November 1998.

HURDER, S. Exotic index theory and the Novikov conjecture. *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 2. London Math. Soc. Lecture Note Ser. 227 (1995), 255–276.

KASPAROV, G. G. Equivariant KK-theory and the Novikov conjecure. *Invent. Math. 91* (1988), 147–201.

—— K-theory, group C^*-algebras, and higher signatures (Conspectus). *Novikov Conjectures, Index Theorems and Rigidity*, Vol. 1. London Math. Soc. Lecture Note Ser. 226. Cambridge Univ. Press (1995), 101–146.

KASPAROV, G. G. and G. SKANDALIS. Groups acting on buildings, operator K-theory and the Novikov conjecture. *K-Theory 4* (1991), 303–337.

—— Groupes "boliques" et conjecture de Novikov. *C. R. Acad. Sci. Paris, Sér. I Math. 319* (1994), 815–820.

MISHCHENKO, A. S. Infinite-dimensional representations of discrete groups, and higher signatures. *Math. USSR Izvestija 8* (1974), 85–111.

NATSUME, T. The Baum-Connes conjecture, the commutator theorem and Rieffel projections. *C. R. Math. Rep. Acad. Sci. Canada 10* (1988), 13–18.

OGLE, C. Assembly maps, K-theory and hyperbolic groups. *K-Theory 6* (1992), 235–265.

ROE, J. Coarse cohomology and index theory on complete Riemannian manifolds. *Mem. Amer. Math. Soc. 104* (1993).

—— Index theory, coarse geometry and topology of manifolds. *CBMS Regional Conference Series in Math. 90*. Amer. Math. Soc., 1996.

ROSENBERG, J. C^*-algebras, positive scalar curvature and the Novikov conjecture. *Publ. Math. I.H.E.S. 58* (1983), 197–212.

ROSENBERG, J. and S. WEINBERGER. An equivariant Novikov conjecture. *K-Theory 4* (1990), 29–53.

WEINBERGER, S. Aspects of the Novikov conjecture. In *Geometric and Topological Invariants of Elliptic Operators. Contemp. Math. 105* (1990), 281–297.

YU, G. L. Coarse Baum-Connes conjecture. *K-Theory 9* (1995), 199–221.

—— Baum-Connes conjecture and coarse geometry. *K-Theory 9* (1995), 223–231.

—— Localization algebras and the coarse Baum-Connes conjecture. *K-Theory 11* (1997), 307–318.

—— The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space. Preprint, July 1998.

9) *Groupoids and foliations*

HECTOR, G. Groupoïdes, feuilletages et C^*-algèbres (quelques aspects de la conjecture de Baum-Connes). *Geometric Study of Foliations (Tokyo, 1993)*, 3–34. World. Sci. Publishing (1994).

MACHO STADLER, M. La conjecture de Baum-Connes pour un feuilletage sans holonomie de codimension un sur une variété fermée. *Publ. Mat. 33* (1989), 445–457.

TAKAI, H. Baum-Connes conjectures and their applications. *World Sci. Adv. Ser. Dyn. Syst. 5* (1987), 89–116.

—— A counterexample to strong Baum-Connes conjectures for foliated manifolds. *World Sci. Adv. Ser. Dyn. Syst. 7* (1989), 149–154.

—— On the Baum-Connes conjecture. In *Mappings of Operator Algebras. Progr. Math. 84*, Birkhäuser (1990), 183–197.

TORPE, A.-M. K-theory for the leaf space of foliations by Reeb components. *J. Funct. Anal. 61* (1985), 15–71.

TU, J.-L. La conjecture de Novikov pour les feuilletages hyperboliques. *K-Theory 16* (1999), 129–184.

—— La conjecture de Baum-Connes pour les feuilletages moyennables. *K-Theory 17* (1999), 215–264.

Paul Baum

Mathematics Department
Pennsylvania State University
University Park, PA 16802
U. S. A.
e-mail: baum@math.psu.edu

Alain Connes

Collège de France
3, rue d'Ulm
F-75005 Paris
and
Institut des Hautes Études Scientifiques
35, route de Chartres
F-91440 Bures-sur-Yvette
France
e-mail: connes@ihes.fr