

## 2.1 Double fibrations of homogeneous spaces

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We briefly recall some classical semisimple notations, as used for instance in Helgason's books. Let  $\mathfrak{a}$  be a maximal abelian subspace of  $\mathfrak{p}$ . Related to the restricted root system of the pair  $(\mathfrak{g}, \mathfrak{a})$  are the eigenspaces  $\mathfrak{g}_\alpha$ , the *Iwasawa decomposition*  $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$  of the Lie algebra and  $G = KAN$  for the group (unique decomposition of each element of  $G$  into a product of factors in the respective subgroups); the subgroups  $A$ , resp.  $N$ , of  $G$  are abelian, resp. nilpotent. The half sum of positive roots (counted with multiplicities) is a linear form  $\rho$  on  $\mathfrak{a}$ ; we write  $a^\rho = e^{\rho(\log a)}$  for  $a \in A$ . Let  $M$ , resp.  $M'$ , denote the centralizer, resp. normalizer, of  $A$  in  $K$ . Then  $W = M'/M$  is a finite group called the Weyl group.

Let  $y_o$  denote the orbit  $N \cdot x_o \subset X$ . The *horocycles* of  $X$  are the submanifolds  $g \cdot y_o$ , for  $g \in G$ . Since  $g \cdot y_o = y_o$  (globally) if and only if  $g \in MN$ , the space of all horocycles is  $Y = G/MN$ .

e. ISOTROPIC RIEMANNIAN SYMMETRIC SPACES. A Riemannian manifold  $X$  is called *isotropic* if, for every  $x \in X$  and every pair of unit tangent vectors  $V, W$  to  $X$  at  $x$ , there exists an isometry of  $X$  leaving  $x$  fixed and mapping  $V$  to  $W$ . The connected isotropic Riemannian manifolds are the Euclidean spaces  $\mathbf{R}^n$ , the *hyperbolic spaces* i.e. the Riemannian symmetric spaces of the noncompact type and of rank one ( $\dim \mathfrak{a} = 1$ ), and their compact analogues, spheres and projective spaces. The compact spaces will not be considered in this paper, so that most of our examples will be taken from the list

$$\mathbf{R}^n, H^n(\mathbf{R}), H^n(\mathbf{C}), H^n(\mathbf{H}), H^{16}(\mathbf{O}).$$

Among them we shall often restrict ourselves to the *classical hyperbolic spaces*  $H^n(\mathbf{F})$ , with  $\mathbf{F} = \mathbf{R}, \mathbf{C}$  or  $\mathbf{H}$ .

## 2. GEOMETRIC SETTING

### 2.1 DOUBLE FIBRATIONS OF HOMOGENEOUS SPACES

The general group-theoretic setting for Radon transforms, introduced by Helgason in the sixties, is motivated by the well-known example of points and hyperplanes in the Euclidean space  $\mathbf{R}^n$ . The set of points and the set of hyperplanes are both homogeneous spaces of the isometry group of  $\mathbf{R}^n$ , and it turns out that the fundamental "incidence" relation (a point  $x$  belongs to a hyperplane  $y$ ), as well as the defining integral of the Radon transform, have simple expressions in terms of Lie groups and invariant measures. This observation suggests considering the following general situation.

Let  $X$  and  $Y$  be two manifolds, with given origins  $x_o \in X$  and  $y_o \in Y$ , and assume a real Lie group  $G$  acts transitively on both manifolds  $X$  and  $Y$ . Two elements  $x \in X$  and  $y \in Y$  are said to be *incident* if there exists some  $g \in G$  such that  $x = g \cdot x_o$  and  $y = g \cdot y_o$ . Roughly speaking, if we think of  $g$  as a motion, this means that  $x$  and  $y$  have the same relative position as the origins  $x_o$  and  $y_o$ .

A more convenient formulation is obtained in terms of the isotropy subgroups  $K$ , resp.  $H$ , of  $x_o$ , resp.  $y_o$ , in  $G$ . They are closed Lie subgroups of  $G$ , and the manifolds  $X, Y$  can be identified with the homogeneous spaces of left cosets  $G/K, G/H$  respectively; in particular we may write  $x_o = K, y_o = H, g \cdot x_o = gK$ , etc. The points  $x = g'K \in X$  and  $y = g''H \in Y$  are then incident if and only if there exists  $g \in G$  such that  $g'K = g \cdot x_o = gK$  and  $g''H = g \cdot y_o = gH$ , in other words if the left cosets  $g'K$  and  $g''H$ , as subsets of  $G$ , are not disjoint (they meet at  $g$ ).

Given  $y = g''H$ , we see that  $x$  is incident to  $y$  if and only if  $x = g'hK$  for some  $h \in H$ . Given  $x = g'K$ , the point  $y$  is incident to  $x$  if and only if  $y = g'kH$  for some  $k \in K$ .

In the above example  $X$ , resp.  $Y$ , is the set of points, resp. hyperplanes, of  $\mathbf{R}^n$  and  $G$  is the group of all isometries. But hyperplanes can also be viewed as subsets of  $X = \mathbf{R}^n$ , and the incidence relation boils down to the familiar “the point  $x$  belongs to the hyperplane  $y$ ” if and only if the chosen origin  $x_o$  belongs to the chosen origin  $y_o$ . Lemma 1 below extends this fact to Riemannian manifolds. More general incidence relations can be considered, however, and will be helpful in Section 6.

Clearly, the group  $G$  acts transitively on the subset  $Z$  of  $X \times Y$  consisting of all incident couples  $(x, y) = (g \cdot x_o, g \cdot y_o)$ , with  $K \cap H$  as the isotropy subgroup of the origin  $(x_o, y_o) \in Z$ . Thus  $Z = G/(K \cap H)$  can be endowed with a structure of manifold, and the present setting can be summarized by the following *double fibration of homogeneous spaces*

$$\begin{array}{ccc} Z = G/(K \cap H) & \subset & X \times Y \\ \downarrow & & \searrow \\ X = G/K & & Y = G/H, \end{array}$$

where the arrows denote the natural projections.

Radon transforms can be studied with more general double fibrations of manifolds  $X, Y, Z$  (without groups), as introduced by Gel'fand et al. [4]. We refer to Guillemin and Sternberg ([6], p.340, 370) for their basic properties; this theory has been developed in several papers by Boman, Quinto, and others.