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Artikel: INVERTING RADON TRANSFORMS : THE GROUP-THEORETIC APPROACH

Kapitel: 6.1 SHIFTS

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6.1 SHIFTS

As before, let $X = G/K$ and $Y = G/H$ be two homogeneous spaces, with K compact, and

$$Ru(g \cdot y_o) = \int_H u(gh \cdot x_o)\, dh$$

be the corresponding Radon transform of $u \in C_c(X)$.

Let $t \in G$ be a "*shift*", fixed at first. Replacing the origin $y_o = H$ in Y by the shifted origin $y_t = t \cdot y_o$, with stabilizer subgroup $H_t = tHt^{-1} \subset G$, we obtain the new identification $Y = G/H_t$, and a new incidence relation between X and Y. A point $x = g \cdot x_o \in X$ is now incident to $y \in Y$ if and only if there exists $\gamma \in G$ such that

$$x = \gamma \cdot x_o \quad \text{and} \quad y = \gamma \cdot y_t = \gamma t \cdot y_o\,,$$

i.e.

$$y = gkt \cdot y_o\,,$$

for some $k \in K$. The corresponding *shifted dual transform* of $v \in C(Y)$ is

$$R_t^* v(g \cdot x_o) = \int_K v(gkt \cdot y_o)\, dk\,.$$

REMARK. We now have two double fibrations

$$\begin{array}{ccc} Z = G/(K \cap H) & & \\ \downarrow & \searrow & \\ X = G/K & & Y = G/H\,, \end{array} \qquad \begin{array}{ccc} Z_t = G/(K \cap H_t) & & \\ \downarrow & \searrow & \\ X = G/K & & Y = G/H_t\,, \end{array}$$

and we are dealing with the Radon tranform R given by the first and the dual transform R_t^* given by the second. The transform R_t associated with the second diagram is

$$R_t u(g \cdot y_o) = \int_H u(ght^{-1} \cdot x_o)\, dh\,;$$

but, excepting the proof of Proposition 12, it will not be used in the sequel.

LEMMA 11. *Let $u \in C_c(X)$ and $g, t \in G$. Then*

$$(R_t^* Ru)(g \cdot x_o) = (Ru_g)(t \cdot y_o)\,,$$

where u_g is the K-invariant function on X defined by

$$u_g(x) = \int_K u(gk \cdot x)\, dk\,.$$

Proof. Immediate, since

$$(R_t^* Ru)(g \cdot x_o) = \int_{K \times H} u(gkth \cdot x_o)\, dk\, dh = \int_H u_g(th \cdot x_o)\, dh\,. \qquad \square$$

Before proceeding we mention the following extension of Proposition 3 to shifted transforms. This result will not be used in the sequel.

PROPOSITION 12. *Let G and H be unimodular, K compact, $X = G/K$ and $Y = G/H$. For any $u \in C_c(X)$ and $t \in G$ we have*

$$R_t^* Ru = u * S_t$$

(convolution on X). Here S_t is the K-invariant distribution on X defined by $S = R_t^* R\delta$, and δ is the Dirac distribution at the origin $x_o = K$ of X, i.e.

$$\langle S_t, u \rangle = R^* R_t u(x_o) = \int_{K \times H} u(kht^{-1} \cdot x_o)\, dk\, dh\,.$$

Proof. The proof of Proposition 3 can be repeated here, with $R^* R_t$ as the dual of $R_t^* R$. The claim can also be checked directly, writing, for $\varphi \in \mathcal{D}(X)$,

$$\langle R_t^* Ru, \varphi \rangle = \int_{G \times H} u(gth \cdot x_o) \varphi(g \cdot x_o)\, dg\, dh\,,$$

and changing variables into $h' = h^{-1}$, $g' = gth$; the result follows easily, G and H being unimodular groups. Details are left to the reader. $\square$

6.2 RADON INVERSION BY SHIFTS

The elementary Lemma 11 can be used in the following way. Assume the transform R can be inverted at the origin for K-invariant functions on X, say

$$u(x_o) = \langle T_{(y)}, Ru(y) \rangle\,, \tag{13}$$

where T is some linear form on a space of functions on Y. Then, replacing u by the K-invariant function u_g in the lemma, we obtain

$$u(g \cdot x_o) = u_g(x_o) = \langle T, Ru_g \rangle\,.$$

The roles of g and t can now be interchanged by Lemma 11, whence

$$u(x) = \langle T_{(t)}, R_t^* Ru(x) \rangle\,, \tag{14}$$

for arbitrary $u \in \mathcal{D}(X)$ and $x \in X$. The notation $T_{(t)}$ means that T now acts on the shift variable t, or $t \cdot y_o$ to be precise. Since $R_{kth}^* Ru(x) = R_t^* Ru(x)$ for $k \in K$ and $h \in H$, this variable may actually be taken in $K \backslash G/H$.