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*The general inversion formula* (14) *for* $R$ *thus follows from the special case* (13) *of* $K$*-invariant functions at the origin*, thanks to the shifted dual transform.

If $X$ is an isotropic space, the above trick (replace $u$ by $u_g$) simply means replacing $u(x)$ by its mean value over the sphere with center $g \cdot x_o$ and radius $d(x_o, x)$.

### 6.3 EXAMPLES

**a.** HOROCYCLE TRANSFORM. We first consider the horocycle Radon transform on $X = G/K$, a Riemannian symmetric space of the noncompact type. Using the classical semisimple notations related to an Iwasawa decomposition $G = KAN$ (see Notations, **d**), we take the point $x_o = K$, resp. the horocycle $y_o = N \cdot x_o$, as the origin in $X$, resp. in $Y = G/MN$. Then

$$Ru(g \cdot y_o) = \int_N u(gn \cdot x_o)\, dn$$

(integrating over $M$ is unnecessary here) and the dual transform shifted by $a \in A$ is

$$R_a^* v(g \cdot x_o) = \int_K v(gka \cdot y_o)\, dk\,.$$

For $K$-invariant $u$ the decomposition $g = kan$ gives

$$Ru(g \cdot y_o) = Ru(a \cdot y_o) = \int_N u(an \cdot x_o)\, dn = a^{-\rho} \mathcal{A}u(a)\,;$$

the *Abel transform* $\mathcal{A}$ is defined by this equality.

For $K$-invariant $u \in \mathcal{D}(X)$ we have $\mathcal{A}u \in \mathcal{D}(A)$. Let $\mathfrak{a}^*$ be the dual space of $\mathfrak{a}$. It is known from spherical harmonic analysis on $X$ that the classical Fourier transform

$$\widehat{\mathcal{A}u}(\lambda) = \int_A a^{-i\lambda} \mathcal{A}u(a)\, da\,, \quad \lambda \in \mathfrak{a}^*\,,$$

coincides with the spherical transform of $u$, with the inversion formula ([9] p. 454)

$$u(x_o) = C \int_{\mathfrak{a}^*} \widehat{\mathcal{A}u}(\lambda)\, |c(\lambda)|^{-2}\, d\lambda\,, \tag{15}$$

where $C$ is a positive constant and $c(\lambda)$ is Harish-Chandra's function. Since

$C \cdot |c(\lambda)|^{-2}$ has polynomial growth on $\mathfrak{a}^*$ its Fourier transform is a tempered distribution $T$ on $A = \exp \mathfrak{a}$ such that

$$u(x_o) = \langle T, \mathcal{A}u \rangle = \langle T_{(a)}, a^\rho Ru(a \cdot y_o) \rangle .$$

Thus $T$ inverts the Abel transform at the origin. By (14) we obtain the next theorem.

THEOREM 13. *Let $X$ be a Riemannian symmetric space of the noncompact type. Its horocycle Radon transform $R$ can be inverted by*

$$u(x) = \langle T_{(a)}, a^\rho R_a^* Ru(x) \rangle, \ x \in X ,$$

*for $u \in \mathcal{D}(X)$. The distribution $T_{(a)}$ (acting on the variable $a \in A$) is, up to a constant factor, the Fourier transform of $|c(\lambda)|^{-2}$.*

REMARKS.

(i) This extends a result by Berenstein and Tarabusi [2] for $X = H^n(\mathbf{R})$, obtained by direct calculations.

(ii) Helgason's original inversion formula ([11], p. 116)

$$u(x) = R^* \Lambda \overline{\Lambda} Ru(x)$$

follows easily from Theorem 13. Indeed Helgason's operator $\Lambda\overline{\Lambda}$ is defined as follows ([11], p. 111). Given $v \in \mathcal{D}(Y)$ and $g = kan \in G$, multiply $v(g \cdot y_o) = v(ka \cdot y_o)$ by $a^\rho$, take the Fourier transform with respect to $a \in A$, multiply it by $C \cdot |c(\lambda)|^{-2}$ (an even function of $\lambda$), take the inverse Fourier transform, and multiply by $a^{-\rho}$; the result is $\Lambda\overline{\Lambda}v(g \cdot y_o)$. In other words

$$\Lambda\overline{\Lambda}v(g \cdot y_o) = \Lambda\overline{\Lambda}v(ka \cdot y_o) = a^{-\rho} \left(T * (a^\rho v)\right)(ka \cdot y_o) ,$$

where $*$ is the convolution on $A$ with respect to $a$. Let $b$ denote a variable in $A$; since $T$ is even we have

$$\begin{aligned} \Lambda\overline{\Lambda}v(g \cdot y_o) &= a^{-\rho} \langle T_{(b)}, (ab)^\rho v(kab \cdot y_o) \rangle \\ &= \langle T_{(b)}, b^\rho v(kab \cdot y_o) \rangle = \langle T_{(b)}, b^\rho v(gb \cdot y_o) \rangle . \end{aligned}$$

Replacing $v$ by $Ru$, $g$ by $gk$ and integrating with respect to $k \in K$ we obtain

$$\begin{aligned} R^* \Lambda\overline{\Lambda} Ru(g \cdot x_o) &= \int_K \langle T_{(b)}, b^\rho Ru(gkb \cdot y_o) \rangle dk \\ &= \left\langle T_{(b)}, b^\rho \int_K Ru(gkb \cdot y_o)\, dk \right\rangle = \langle T_{(b)}, b^\rho R_b^* Ru(g \cdot x_o) \rangle . \end{aligned}$$

By Theorem 13 this is $u(g \cdot x_o)$, as claimed.

(iii) Note that $T$ is supported at the origin if and only if $|c(\lambda)|^{-2}$ is a polynomial, i.e. if the Lie algebra $\mathfrak{g}$ has only one conjugacy class of Cartan subalgebras (see Corollary 20 below).

**b.** TOTALLY GEODESIC TRANSFORM ON CLASSICAL HYPERBOLIC SPACES. We retain the notation of Section 4.3 **c**.

THEOREM 14. *Let $X = H^m(\mathbf{F})$, $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$, be one of the classical hyperbolic spaces, let $\mathfrak{s}$ be any $\mathbf{F}$-vector subspace of $\mathfrak{p} = \mathbf{F}^m$, and $T$ any unit vector orthogonal to $\mathfrak{s}$ in $\mathfrak{p}$.*

*For the Radon transform defined by the totally geodesic submanifolds $y = g \cdot \operatorname{Exp} \mathfrak{s}$, of (real) dimension $d$, we have the following inversion formulas by means of shifted dual transforms, for $u \in \mathcal{D}(X)$ and $x \in X$.*

(i) *If $d = 2k+1$ is odd, $k \geq 0$,*

$$2^k \pi^{k+1} u(x) = \left(\sigma^{-1}\partial_\sigma\right)^{k+1} \int_0^\sigma \left(R^*_{\exp t(\tau)T} Ru(x)\right) (\sigma^2 - \tau^2)^{-1/2} d\tau \Big|_{\sigma=1},$$

*where $t(\tau)$ denotes the positive solution of the equation* $\cosh t = 1/\tau$.

(ii) *If $d = 2k$ is even, $k \geq 1$, there exists a polynomial of degree $k$*

$$Q_k(\lambda) = \frac{2^k k!}{(2k)!}\lambda^k + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1),$$

*with rational coefficients (depending on $k$ and $q' = \dim \mathfrak{s}_{2\alpha}$), such that*

$$(-2\pi)^k u(x) = Q_k(\partial_t^2)\left(R^*_{\exp tT} Ru(x)\right)_{t=0}.$$

REMARKS. This extends a result proved by Helgason ([10], p. 144, or [14], p. 97) for $\mathbf{F} = \mathbf{R}$. In case (i), a look at the proof below shows that an arbitrary positive integer $\ell$ may be added to the exponents of $\sigma^{-1}\partial_\sigma$ and $\sigma^2 - \tau^2$; Helgason's result is obtained for $\ell = k$. From the proof of case (ii) we obtain for $k = 1, 2$

$$Q_1(\partial_t^2) = \partial_t^2 + q' + 1$$

$$Q_2(\partial_t^2) = \frac{1}{3}\partial_t^4 + \left(2q' + \frac{14}{3}\right)\partial_t^2 + (q'+1)(q'+3).$$

Our $d$ is of course even whenever $\mathbf{F} = \mathbf{C}$ or $\mathbf{H}$. A comparison with Section 4.3 **c** shows that (except for $\mathbf{F} = \mathbf{R}$) the present assumption on $\mathfrak{s}$ is stronger than in Theorem 8.

*Proof of Theorem 14.* In order to use spherical coordinates on totally geodesic submanifolds of $X$, we need a lemma. As in Section 4.3 **c**, the matrices in $\mathfrak{p}$ can be identified to vectors $V = (V_1, \ldots, V_m) \in \mathbf{F}^m$, and the scalar product of $T, V \in \mathfrak{p}$ is

$$(T, V) = \operatorname{Re}\left(\bar{T} \cdot V\right), \quad \text{with } \bar{T} \cdot V = \sum_{i=1}^{m} \bar{T}_i V_i .$$

Let $\| \ \|$ be the corresponding norm.

LEMMA 15. *Let $X = H^m(\mathbf{F})$ be a classical hyperbolic space.*

(i) *Let $T, V \in \mathfrak{p}$. In the geodesic triangle with vertices $x_o$ (the origin of $X$), $\operatorname{Exp} T$ and $\exp T \cdot \operatorname{Exp} V$, the Riemannian lengths of the sides are $t = \|T\|$, $r = \|V\|$ and $w$ given by*

$$\cosh^2 w = \left(\cosh t \cosh r + \frac{\sinh t}{t}\frac{\sinh r}{r}(T, V)\right)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}\left|\bar{T} \cdot V - (T, V)\right|\right)^2 .$$

(ii) *Let $\mathfrak{s} \subset \mathfrak{p}$ be a Lie triple system. If $T \in \mathfrak{p}$ is orthogonal to $\mathfrak{s}$, the totally geodesic submanifold $\exp T \cdot \operatorname{Exp} \mathfrak{s}$ is at distance $t = \|T\|$ from the origin.*

*Proof.* (i) The Riemannian distance from $x_o$ to $\operatorname{Exp} T$ is $\|T\| = t$. Transforming $x_o$ and $\operatorname{Exp} V$ by the isometry $\exp T \in G$ shows that the second side of the triangle has length $r$. The third side is $w = \|W\|$, where $W$ is the unique $W \in \mathfrak{p}$ such that $\operatorname{Exp} W = \exp T \cdot \operatorname{Exp} V$, in other words

$$\exp W = (\exp T \exp V)\, k$$

for some $k \in K$. The map $g \mapsto g\theta(g)^{-1}$, where $\theta$ is the Cartan involution of $G$, transforms this equality into

$$\exp 2W = \exp T \exp 2V \exp T .$$

By elementary matrix computations $T^3 = t^2 T$, and the exponential is

$$\exp T = I + \frac{\sinh t}{t} T + \frac{\cosh t - 1}{t^2} T^2 ,$$

where $I$ is the unit matrix. Now $\operatorname{tr} T = 0$ and $\operatorname{tr} T^2 = 2t^2$ is real, so that taking the traces we obtain

$$\operatorname{tr}(\exp 2W) = \operatorname{Re}\operatorname{tr}(\exp 2W) = \operatorname{Re}\operatorname{tr}(\exp 2T \exp 2V) ;$$

indeed $\operatorname{Re}\operatorname{tr}(gg') = \operatorname{Re}\operatorname{tr}(g'g)$ for $g, g' \in G$, even when $\mathbf{F} = \mathbf{H}$.

Taking account of

$$\operatorname{Re}\operatorname{tr} TV = 2(T,V)\,, \qquad \operatorname{tr} T^2V = \operatorname{tr} TV^2 = 0\,,$$
$$\operatorname{Re}\operatorname{tr} T^2V^2 = t^2r^2 + \left|\overline{T}\cdot V\right|^2,$$

the expression of $\cosh w$ follows after some elementary calculations.

(ii) Let $y = \exp T \cdot \operatorname{Exp} \mathfrak{s}$. By (i) with $V \in \mathfrak{s}$ and $(T,V) = 0$, the distance $w$ of the origin to the point $\operatorname{Exp} W = \exp T \cdot \operatorname{Exp} V$ of $y$ is given by

$$\cosh^2 w = (\cosh t \cosh r)^2 + \left(\frac{\sinh t}{t}\frac{\sinh r}{r}\left|\overline{T}\cdot V\right|\right)^2.$$

Therefore $w \geq t$, with equality if and only if $V = 0$, and $\operatorname{Exp} T$ is the unique point of $y$ closest to $x_o$ (geodesic orthogonal projection of the origin on $y$). The lemma is proved. $\square$

Going back to Theorem 14, let $g \in G$ and let $y = g \cdot \operatorname{Exp} \mathfrak{s}$ be an arbitrary given totally geodesic submanifold, element of $Y$. The minimum distance between $y$ and the origin $x_o$ is obtained at a point $\operatorname{Exp} T \in y$, with $T \in \mathfrak{p}$. In particular there exists $V \in \mathfrak{s}$ such that $\operatorname{Exp} T = g \cdot \operatorname{Exp} V$, i.e. $(\exp T)k = g \exp V$ for some $k \in K$. But $\operatorname{Exp} \mathfrak{s}$ is globally invariant under the action of $\exp V$, so that $y = (\exp T)k \cdot \operatorname{Exp} \mathfrak{s} = \exp T \cdot \operatorname{Exp}(k\cdot\mathfrak{s})$. Changing notation, we may write $\mathfrak{s}$ for $k\cdot\mathfrak{s}$ and $y = \exp T \cdot \operatorname{Exp} \mathfrak{s}$.

Let $V \in \mathfrak{s}$. On the geodesic $\exp T \cdot \operatorname{Exp} sV$, $s \in \mathbf{R}$, contained in $y$, the minimum distance to $x_o$ is obtained for $s = 0$. By Lemma 15 (i) with $sV$ instead of $V$, this implies $(T,V) = 0$ so that $T$ is orthogonal to $\mathfrak{s}$ and Lemma 15 (ii) applies.

Besides, if we assume $\mathfrak{s}$ is a $\mathbf{F}$-vector subspace of $\mathfrak{p}$ therefore a Lie triple system (Section 4.3 **c**), the vector $T$ must be orthogonal to all $V\lambda$, $V \in \mathfrak{s}$, $\lambda \in \mathbf{F}$, whence $\overline{T}\cdot V = 0$. By Lemma 15 the distance $w = w(t,r)$ between $x_o$ and an arbitrary point $x = \exp T \cdot \operatorname{Exp} V$ of $y$ is simply given by

(16) $$\cosh w(t,r) = \cosh t \cosh r\,, \quad t = \|T\|\,, \quad r = \|V\|\,,$$

the same expression as for real hyperbolic spaces.

According to (13) and (14) we only need to invert $R$ at the origin for a $K$-invariant function $u$. As shown in Section 4.1 **a**, Lemma 1 applies and $Ru(y) = \int_y u(x)\,dm_y$. When $u$ is radial the integral can be obtained in spherical coordinates on $y$ with origin $\operatorname{Exp} T$, as

(17) $$Ru(y) = \int_0^\infty u(w(t,r))A_o(r)\,dr$$

where $A_o(r) = \omega_d(\sinh r)^{d-1}(\cosh r)^{q'}$ is the area of spheres of radius $r$ in $\mathcal{Y}$. By (16) and (17) $Ru$ may be viewed as a smooth even function $Ru(t)$ of $t \in \mathbf{R}$.

The end of the proof is now similar to the case of $H^n(\mathbf{R})$, as given in [11], p. 53 or [14], p. 97. Let $\tau = (\cosh t)^{-1}$, and let $t = t(\tau) \geq 0$ denote the inverse function. Introducing the functions

$$\varphi(\tau) = \tau^{-d-q'} u(t(\tau)), \quad \psi(\tau) = \tau^{-1-q'} Ru(t(\tau)),$$

which are $C^\infty$ on $]0, 1]$, (17) becomes

$$\psi(\tau) = \omega_d \int_0^\tau \varphi(\rho) \left(\tau^2 - \rho^2\right)^{(d/2)-1} d\rho. \tag{18}$$

*Proof of* (i). The Abel type integral equation (18) can be inverted as usual: it implies that, for any $a > 0$, $\sigma > 0$,

$$\Gamma\left(\frac{d}{2} + a\right) \int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{a-1} \tau d\tau =$$

$$= \pi^{d/2}\Gamma(a) \int_0^\sigma \varphi(\rho) \left(\sigma^2 - \rho^2\right)^{(d/2)+a-1} d\rho$$

and, choosing $a > 0$ such that $N = (d/2) + a$ is a strictly positive integer, it follows easily that

$$2^{N-1}\pi^{d/2}\Gamma(a)\varphi(\sigma) = \sigma \left(\sigma^{-1}\partial_\sigma\right)^N \left(\int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{a-1} \tau d\tau\right).$$

If $d = 2k + 1$ is odd, $k \geq 0$, the smallest such $a$ is $1/2$ so that $N = k + 1$ and

$$2^k \pi^{k+1} \varphi(\sigma) = \sigma \left(\sigma^{-1}\partial_\sigma\right)^{k+1} \left(\int_0^\sigma \psi(\tau)(\sigma^2 - \tau^2)^{-1/2} \tau d\tau\right), \quad \sigma > 0;$$

the derivatives cannot be taken here under the integral. Besides $d$ can only be odd for $\mathbf{F} = \mathbf{R}$ according to the assumption on $\mathfrak{s}$, and $q' = 0$ in that case. Going back to $u$ and $Ru$ we thus obtain for $\sigma = 1$

$$2^k \pi^{k+1} u(x_o) = \left(\sigma^{-1}\partial_\sigma\right)^{k+1} \int_0^\sigma Ru(t(\tau))(\sigma^2 - \tau^2)^{-1/2} d\tau \Big|_{\sigma=1},$$

for any $K$-invariant $u \in \mathcal{D}(X)$. The claim follows by Section 6.2.

*Proof of* (ii). If $d = 2k$ is even, $k \geq 1$, the integral equation (18) can be directly solved as

$$(2\pi)^k \varphi(\tau) = \tau \left(\tau^{-1}\partial_\tau\right)^k \psi(\tau), \quad \tau > 0.$$

In particular, at the origin,

$$\begin{aligned}(2\pi)^k u(x_o) &= \left(\tau^{-1}\partial_\tau\right)^k \left(\tau^{-1-q'} Ru(t(\tau))\right)_{\tau=1}\\ &= \left(\partial_\tau^k + \cdots + (-1)^k (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t(\tau))\Big|_{\tau=1}.\end{aligned}$$

To switch over to derivatives with respect to $t$ we note that, if $g(\tau) = f(t)$ with $\tau = (\cosh t)^{-1} = 1 - \frac{t^2}{2} + \cdots$, identification of Taylor expansions at $\tau = 1$, resp. $t = 0$, leads to

$$\left(-\frac{1}{2}\right)^k \frac{g^{(k)}(1)}{k!} = \frac{f^{(2k)}(0)}{(2k)!} + \cdots + a_k f''(0),$$

where dots are a sum of even derivatives of $f$ multiplied by some rational coefficients (like $a_k$). Therefore

$$(-2\pi)^k u(x_o) = \left(\frac{2^k k!}{(2k)!}\partial_t^{2k} + \cdots + (q'+1)(q'+3)\cdots(q'+2k-1)\right) Ru(t)|_{t=0},$$

for any $K$-invariant $u \in \mathcal{D}(X)$, whence the claim by Section 6.2. $\square$

## 6.4 The amusing formula generalized

**a.** To motivate the forthcoming generalizations of the amusing formula (12) and their applications to Radon inversion, we briefly recall the classical example of points and hyperplanes in the Euclidean space $X = \mathbf{R}^n$. Let $(\omega, p)$ be parameters for the hyperplane defined by the equation $\omega \cdot x = p$, where $\omega$ is a unit vector, $p$ is a real number and $\cdot$ is the scalar product. Given $t \in \mathbf{R}$ and a point $x \in \mathbf{R}^n$, the parameters $(\omega, p) = (\omega, t + \omega \cdot x)$ define a hyperplane at distance $|t|$ from $x$, and

$$R_t^* v(x) = \int_{S^{n-1}} v(\omega, t + \omega \cdot x)\, d\omega$$

is the corresponding shifted dual Radon transform, where $v(\omega, p) = v(-\omega, -p)$ is an arbitrary smooth even function on $S^{n-1} \times \mathbf{R}$. Changing $\omega$ into $-\omega$ in the integral shows that $R_t^* v(x)$ is an even function of $t$.

Since $\sum \omega_i^2 = 1$ it is easily checked that

$$\left(\partial_t^2 - \Delta_x\right) v(\omega, t + \omega \cdot x) = 0,$$

where $\Delta_x$ is the Euclidean Laplace operator acting on $x$. Thus $R_t^* v(x)$, as a function of $(x, t)$ in $\mathbf{R}^n \times \mathbf{R}$, is a solution of the wave equation, being an