VITALI'S CONVERGENCE THEOREM ON TERM BY TERM INTEGRATION

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1. INTRODUCTION

In this article we discuss a convergence theorem of Vitali [29] which appeared in 1907, before Lebesgue proved the dominated convergence theorem. This theorem is in some ways stronger than the standard convergence theorems, and deserves to be better known than it is. Vitali proves that if a sequence of integrable functions f_n converges a.e. to an integrable function f (on a space of finite measure), then the integrals of f_n on any measurable subset converge to those of f, if and only if the integrals are uniformly absolutely continuous. The hard part is to show that convergence of the integrals on any measurable subset implies uniform absolute continuity. Subsequently (in 1915) de la Vallée Poussin [27] simplified Vitali's proof (this is also not well-known), and in 1922 Hahn [10] was led to prove the much better known Vitali-Hahn-Saks Theorem. If Vitali's paper is quoted today, it is usually either as (i) a forerunner to the Vitali-Hahn-Saks Theorem, or (ii) as the much weaker result that L^1 convergence is equivalent to uniform absolute continuity. Note that Vitali's result shows that when f_n converges a.e. to f, weak convergence implies strong convergence. We give here, in modern language and notation, Vitali's original proof, de la Vallée Poussin's simplification, and finally, Hahn's original proof of the Vitali-Hahn-Saks Theorem. This last is also not well-known, having been superseded by the Baire category proof of Saks [25] (and Banach [1]). This article is not directed to experts in the history of the subject, but to the vast majority of real analysts, who though they teach the subject, are not aware of the history or existence of these proofs. Numbers in square brackets refer to the reference list at the end of the article.

The three standard convergence theorems are:

1. THE MONOTONE CONVERGENCE THEOREM (MCT). *If* $\{f_n\}$ *is a sequence of non negative measurable functions on a measurable set* E *with* $f_n \leq f_{n+1}$ *for all* $n \in \mathbf{N}$, *then* $\int_E \lim f_n = \lim \int_E f_n$.

2. FATOU'S LEMMA (Fatou). *If* $\{f_n\}$ *is a sequence of non negative measurable functions on a measurable set* E, *then* $\int_E \liminf f_n \leq \liminf \int_E f_n$.

[The non-negativity in Fatou and the inequalities in MCT may hold a.e.]

3. THE DOMINATED CONVERGENCE THEOREM (DCT). *If* $\{f_n\}$ *is a sequence of measurable functions on a measurable set* E *such that* $f_n \to f$ *a.e. on* E *and if there exists a function* g, *integrable on* E *with* $|f_n| \leq g$ *a.e. on* E, *then* $\int_E f = \lim \int_E f_n$.

A special case is

THE BOUNDED CONVERGENCE THEOREM (BCT): *If* E *is a set of finite measure,* $\{f_n\}$ *measurable on* E *such that* $f_n \to f$ *a.e. on* E, *and if there exists a real number* $M > 0$ *with* $|f_n| \leq M$ *a.e. on* E, *then* $\int_E f = \lim \int_E f_n$.

They are most usually proved in this order; sometimes BCT is proved first. Other convergence results involving mean convergence come somewhat later. DCT is the result most often used in practice (though MCT has perhaps a deeper theoretical significance: see below).

Historically, things were very different. Lebesgue's thesis [14] (referred to as *'thèse'* in what follows) appeared in 1902 as a paper in *Annali di Matematica* entitled «Intégrale, longueur, aire». Lebesgue's main interests were in various ways of constructing the integral, or primitive, in differentiation and the fundamental theorem of the calculus, and in completing the study of measure, initiated and carried quite far by E. Borel. Convergence theorems were not his main interest, and the thesis contains only BCT on p. 259.

Sketch of his proof. Let $\varepsilon > 0$ and let $F_n = \bigcup_{j \geq n}\{|f_j - f| \geq \varepsilon\}$. Then $m(F_n) \to 0$, since $f_n \to f$ a.e. ($m(F_n)$ denotes the measure of F_n). If $E_n = E \backslash F_n$, then

$$\left|\int_E f_n - \int_E f\right| \leq \int_E |f_n - f| = \int_{E_n} |f_n - f| + \int_{F_n} |f_n - f|$$
$$\leq \varepsilon m(E) + 2Mm(F_n)$$
$$< \varepsilon m(E) + 2\varepsilon M \text{ for sufficiently large } n.$$

Lebesgue lectured on his new work at the Collège de France in 1902–03, and these lectures were published as a book entitled *« Leçons sur l'intégration et la recherche des fonctions primitives »* (1st edition 1904 [15], referred to as *Leçons I*). Again, the only convergence theorem proved is BCT, on p. 114. But in the last chapter (Chapter VII) of *Leçons I* (Chapter VII also in the revised 2nd edition of 1928 [19], referred to as *Leçons II*), Lebesgue states the « Problème de l'intégration », six properties which an integral on a suitable class of bounded functions should possess. Property (6) is the convergence property: for $f_n, f \geq 0$ and bounded, if $f_n \leq f_{n+1}$ for all n, and $f_n \uparrow f$ then $\int f_n \to \int f$. Of course this restricted version of MCT for bounded functions follows at once from BCT. [The best historical account of the theory of integration up to 1910 is in Hawkins [12].]

In 1906 (four years after Lebesgue's thesis was published) Beppo Levi [20] proved MCT and independently Fatou [7] proved his lemma. Levi's paper is short and crystal clear, even if your Italian is rudimentary !

Sketch of his proof. Let $0 \leq f_n \leq f_{n+1}$ and $f = \lim f_n$ on E. Assume $m(E) < \infty$. Let $f^k = \min(f, k)$, $f_n^k = \min(f_n, k)$, $k \in \mathbf{N}$, and let $a_k = \int_E f^k$, $a_{n,k} = \int_E f_n^k$. Then $a_{n,k}$ is increasing in n for fixed k, and increasing in k for fixed n, so $\lim_n \lim_k a_{n,k} = \lim_k \lim_n a_{n,k}$ regardless of whether the limits are finite or infinite. Since f_n^k, f^k are bounded by k (and of course ≥ 0), BCT gives $\lim_n a_{n,k} = a_k = \int_E f^k$ for each k. If f and so f_n are integrable, then $\lim_k a_k = \int_E f$, $\lim_k a_{n,k} = \int_E f_n$, and so $\int_E f = \int_E \lim f_n = \lim \int_E f_n$, by the equality of the repeated limits. This happens if either repeated limit is finite, in particular if $\lim_n \int_E f_n$ is finite. If not, both repeated limits are infinite and $\lim_n \int_E f_n = +\infty$.

Fatou's proof of his lemma is very similar. It should be noted that Fatou's long paper is one of the most important of the century. For the first time the new theory of integration is applied to complex function theory; there are also fundamental applications to trigonometric series.

It is not until 1908, that DCT first appears in Lebesgue (1908) p. 9–10 [16] with a sketch of the proof; the same thing happens in Lebesgue (1909) [17] at the top of p. 50. In these papers Lebesgue seeks to apply his new results and finds BCT insufficient. In Lebesgue (1910) [18], in § 15 on page 375, the proof of DCT is given in more detail, still on a set of finite measure.

Sketch of his proof. Let $\varepsilon > 0$; since g is integrable on E, there exists a number $M > 0$, such that $\int_F g < \varepsilon$, where $F = \{g > M\}$; then $\int_F |f_n - f| < 2\varepsilon$, and on $E \setminus F$, the result follows by BCT.

Note that all the theorems so far have been stated and proved for sets E of *finite* measure. There does not seem at that time to have been much interest on anyone's part in extending the results and proofs for the case $m(E) = +\infty$. However, (excluding of course BCT) this is easily done.

2. Vitali's convergence theorem

In 1907, *before* Lebesgue announced DCT, there appeared a remarkable paper by G. Vitali [29], which, I feel, has not received its due, even from Hawkins. In it Vitali proves the following result:

Let E be a set of finite measure (finiteness is essential here). Let $\{f_n\}$ be a sequence of integrable functions such that $f_n \to f$ a.e. with f finite a.e. Then f is integrable and $\int_F f_n \to \int_F f$, for every measurable subset F of E, if and only if the integrals $\int_A f_n$ are uniformly absolutely continuous (uniformly in n): given $\varepsilon > 0$, there exists $\delta > 0$, such that if $m(A) < \delta$, then $\left|\int_A f_n\right| < \varepsilon$ for all n.

This implies that $\int_A |f_n| < 2\varepsilon$. Vitali calls this *equi-absolutely continuous.* Note that this result generalizes at once to any finite measure space.

Vitali first proves that uniform absolute continuity is sufficient for $\int_F f_n \to \int_F f$ for all measurable subsets F of E.

Sketch of his proof. For $h \in \mathbf{N}$, let $G_h = \{|f_n| > 2^h \text{ for some } n\}$. If $\Gamma_h = E \backslash G_h$, then the sequence $\{f_n\}$ is uniformly bounded on Γ_h for every $h \in \mathbf{N}$. Thus on all measurable subsets of Γ_h, convergence of integrals follows by BCT. On the other hand, G_h is a decreasing sequence of sets and $m(\bigcap_h G_h) = 0$, so $m(G_h) \downarrow 0$. So for all sufficiently large h, the uniform absolute continuity condition implies that the contribution of the integrals over G_h is small.

Vitali next proves necessity of the uniform absolute continuity condition when the functions f_n are all non-negative.

Sketch of his proof. If $\int f_n$ are not uniformly absolutely continuous, then for some $\varepsilon > 0$, there exists for each $\delta > 0$, a measurable set F with $m(F) < \delta$ and $n \in \mathbf{N}$ with $\int_F f_n > \varepsilon$. Let $\delta_i > 0$ with $\Sigma \delta_i < \infty$. For each δ_i, there exists a measurable set $G_i \subseteq E$ and $n_i \in \mathbf{N}$ such that $m(G_i) < \delta_i$ and $\int_{G_i} f_{n_i} > \varepsilon$. Let $\Gamma_r = \bigcup_{i=r}^{\infty} G_i$. Then Γ_r decrease with r and $m(\Gamma_r) < \sum_{i=r}^{\infty} \delta_i \to 0$ as $r \to \infty$. For all $i \geq r$, $\int_{\Gamma_r} f_{n_i} > \varepsilon$. Since $f_{n_i} \to f$ a.e. as $i \to \infty$, $\int_{\Gamma_r} f_{n_i} \to \int_{\Gamma_r} f$ by hypothesis, and so for each r, $\int_{\Gamma_r} f \geq \varepsilon > 0$. Put $\Gamma = \bigcap_{r=1}^{\infty} \Gamma_r$. Then $m(\Gamma) = 0$, but $\int_{\Gamma} f \geq \varepsilon > 0$. Contradiction.

Finally Vitali proves necessity in the general case. If $f_n \to f$ a.e. on E and f_n are *completely integrable* on E, i.e. $\int_F f_n \to \int_F f$ for every measurable subset F of E, then $\int f_n$ are uniformly absolutely continuous, i.e. given $\varepsilon > 0$ there exists $\delta > 0$, such that if $m(A) < \delta$, then $\left|\int_A f_n\right| < \varepsilon$ for all $n \in \mathbf{N}$. (This is the deepest and hardest part of Vitali's paper, and is in a sense 'new' even after 93 years !)

Proof (Vitali). All the sets that occur in this proof will be measurable, even when this is not explicitly stated.

STEP I. If $f_n \geq 0$ for all $n \in \mathbf{N}$, we have already seen above that the result is true.

STEP II. Suppose now that $f > 0$ a.e. on E; we can assume that $f > 0$ on *all* of E. Note first that if $f_n \to f$ boundedly then BCT implies that $\int_E |f_n - f| \to 0$ as $n \to \infty$, and so given $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $F \subseteq E$ and $n \geq N$, we have $\left| \int_F f_n - \int_F f \right| < \varepsilon$.

Let $G_n = \{0 < f_j < 2^n, \ \forall j \geq n\}$. $G_n \subseteq G_{n+1}$ for all n and $E = \bigcup_{n=1}^{\infty} G_n$. So $E \backslash G_n \downarrow \varnothing$ and $m(E \backslash G_n) \downarrow 0$.

Now given $\sigma > 0$, there exists $m(n, \sigma)$ such that if $\Gamma_n \subseteq G_n$ then $\left| \int_{\Gamma_n} f_j - \int_{\Gamma_n} f \right| < \sigma$ for all $j \geq m(n, \sigma)$. Let $\varepsilon_n > 0$, $\varepsilon_n \downarrow 0$. We can find a strictly increasing sequence of positive integers $n_1 < n_2 < \cdots < n_i < \cdots$ such that for every subset Γ_{n_i} of G_{n_i},

$$(\dagger) \qquad \left| \int_{\Gamma_{n_i}} f_j - \int_{\Gamma_{n_i}} f \right| < \varepsilon_i \quad \text{for } j \geq n_{i+1} .$$

For every positive integer $n > n_2$, there exists a unique $i \in \mathbf{N}$ such that $n_{i+1} \leq n < n_{i+2}$. For such n, put

$$g_n(x) = \begin{cases} f_n(x) & \text{if } x \in G_{n_i} \\ 0 & \text{if } x \in E \backslash G_{n_i} . \end{cases}$$

Let Γ be a subset of E and $\Gamma_{n_i} = G_{n_i} \cap \Gamma$. For fixed i,

$$\lim_{k \to \infty} \int_{\Gamma_{n_i}} g_k = \lim_{k \to \infty} \int_{\Gamma_{n_i}} f_k = \int_{\Gamma_{n_i}} f .$$

If $n_{i+1} \leq n < n_{i+2}$, then $\int_{\Gamma_{n_i}} g_n = \int_\Gamma g_n$, and so, using ($\dagger$),

$$\lim_{n \to \infty} \int_\Gamma g_n = \lim_{i \to \infty} \int_{\Gamma_{n_i}} f = \int_\Gamma f = \int_\Gamma \lim g_n .$$

Thus g_n are completely integrable on E and since $g_n \geq 0$ for all n, it follows by Step I that $\int g_n$ are uniformly absolutely continuous on E. Put $\phi_n = f_n - g_n$. The ϕ_n are completely integrable on E. To complete the proof of Step II we must show that $\int \phi_n$ are uniformly absolutely continuous.

Observe that if $n_{i+1} \le n < n_{i+2}$, then $\phi_n(x) = 0$ for all $x \in G_{n_i}$. So for all $x \in E$, $\lim \phi_n(x) = 0$, and for any measurable subset $\Omega \subset E$, $\int_\Omega \phi_n \to 0$.

Suppose that $\int \phi_n$ are *not* uniformly absolutely continuous. Then there exists $\sigma > 0$, such that for all $\mu > 0$ and $N \in \mathbf{N}$, there exist $\Gamma \subseteq E$ with $m(\Gamma) < \mu$ and $n > N$ such that $\left|\int_\Gamma \phi_n\right| > \sigma$. Let $\eta_1, \eta_2, \ldots$ be > 0, and such that $\Sigma \eta_i < \frac{\sigma}{2}$. Let Γ_1 be a subset of E for which there exists $t_1 \in \mathbf{N}$ with $\left|\int_{\Gamma_1} \phi_{t_1}\right| > \sigma$. Since $\lim_{i\to\infty} \int_{G_{n_i} \cap \Gamma_1} \phi_{t_1} = \int_{\Gamma_1} \phi_{t_1}$, we can find $i_1 \in \mathbf{N}$ such that $\left|\int_{G_{n_{i_1}} \cap \Gamma_1} \phi_{t_1}\right| > \sigma$. Now there exists $\mu_1 > 0$ such that if $m(\Gamma) < \mu_1$, then $\left|\int_\Gamma \phi_{t_1}\right| < \eta_1$. By our assumption, there exists Γ_2 such that $m(\Gamma_2) < \mu_1$ and $t_2 \ge n_{i_1+1}$ such that $\left|\int_{\Gamma_2} \phi_{t_2}\right| > \sigma$. By the same reasoning, there exists i_2 (necessarily $> i_1$) such that $\left|\int_{G_{n_{i_2}} \cap \Gamma_2} \phi_{t_2}\right| > \sigma$.

Now there exists $\mu_2 > 0$ such that if $m(\Gamma) < \mu_2$, then

$$\left|\int_\Gamma \phi_{t_j}\right| < \eta_2, \ j = 1, 2.$$

There exist a subset Γ_3 with $m(\Gamma_3) < \mu_2$ and $t_3 \ge n_{i_2+1}$, such that $\left|\int_{\Gamma_3} \phi_{t_3}\right| > \sigma$. Again we can find $i_3 \in \mathbf{N}$ such that $\left|\int_{G_{n_{i_3}} \cap \Gamma_3} \phi_{t_3}\right| > \sigma$. Now there exists $\mu_3 > 0$ such that if $m(\Gamma) < \mu_3$, then

$$\left|\int_\Gamma \phi_{t_j}\right| < \eta_3, \ j = 1, 2, 3.$$

Continue in this way to obtain an increasing sequence $i_j, \mu_j > 0$, and $t_j \ge n_{i_{j-1}+1}$ and sets Γ_j such that $m(\Gamma_j) < \mu_{j-1}$ and $\left|\int_{\Gamma_j} \phi_{t_k}\right| < \eta_{j-1}$, $k = 1, 2, \ldots, j-1$, but $\left|\int_{G_{n_{i_j}} \cap \Gamma_j} \phi_{t_j}\right| > \sigma$. Let

$$\Omega_j = \{x \in G_{n_{i_j}} \cap \Gamma_j \text{ such that } \phi_{t_j}(x) \ne 0\}.$$

We claim the sets Ω_j are disjoint. Since $t_j \geq n_{i_{j-1}+1}$ we have $\phi_{t_j} = 0$ on $G_{n_{i_{j-1}}}$, and so $\phi_{t_{j+1}} = 0$ on $G_{n_{i_j}}$. So $\Omega_{j+1} \cap G_{n_{i_j}} = \varnothing$. Since G_n increase with n, i.e. $G_n \subseteq G_{n+1}$ for all n, it follows that $\Omega_{j+1} \cap \Omega_k = \varnothing$ if $k = 1, 2, \ldots, j$, and so Ω_j are all disjoint. Put $\Omega = \bigcup_{j=1}^{\infty} \Omega_j$. Then since $\phi_{t_j} = 0$ on $\Omega_1, \ldots, \Omega_{j-1}$,

$$\int_{\Omega} \phi_{t_j} = \int_{\Omega_j} \phi_{t_j} + \sum_{h=1}^{\infty} \int_{\Omega_{j+h}} \phi_{t_j}.$$

But $\left| \int_{\Omega_j} \phi_{t_j} \right| > \sigma$ and $\left| \int_{\Omega_{j+h}} \phi_{t_j} \right| < \eta_{j+h-1}$, so $\left| \sum_{h=1}^{\infty} \int_{\Omega_{j+h}} \phi_{t_j} \right| < \frac{\sigma}{2}$. Hence $\left| \int_{\Omega} \phi_{t_j} \right| > \frac{\sigma}{2}$ for all j. But since ϕ_n are completely integrable and $\phi_n(x) \to 0$ for all $x \in E$ it follows that $\int_{\Omega} \phi_{t_j} \to 0$ as $j \to \infty$. This is a contradiction, completing the proof of Step II.

Step III. In the general case, put $g_n = f_n - f + 1$. Then $g_n \to 1$ a.e. on E, and g_n is completely integrable. Hence as seen in Step II, the result is true for the g_n, i.e., $\int g_n$ are uniformly absolutely continuous. Hence the same holds for $\int f_n$, completing the proof.

Comments

The concept of complete integrability of a sequence is weaker than weak sequential convergence in L^1. Vitali was of course, in 1907, unaware of L^1 convergence (strong or weak) and its significance. Using Vitali's proof that complete integrability implies uniform absolute continuity, we see that it also implies convergence in L^1. But a *direct* proof that complete integrability implies L^1 convergence (without using Vitali's result) seems hard.

A feature of this remarkable paper is that all the results are stated and proved in terms of series of functions rather than sequences; thus one has to realize that a "series all of whose partial sums are non-negative" corresponds to a sequence of non-negative terms, and is *not* to be confused with "a series of non-negative terms", which corresponds, of course, to an increasing sequence of non-negative terms! Vitali's theorem is obviously a generalization of BCT. He remarks that Beppo Levi's MCT follows from it. Of course, so also does DCT, but Vitali did not know about DCT at the time.

The history and use of this result between 1907 and 1939 is something I would like to know more about! Hawkins [12] mentions the paper, but quotes only what I consider to be a much less important result at the end of the paper. In a footnote on p. 50 of his 1909 paper [17], Lebesgue says that DCT and MCT are special cases of Vitali's convergence theorem. He also states that DCT can be extended to sets of infinite measure. On p. 365 of the 1910 paper [18] he again refers to Vitali's Theorem, saying that it gives a necessary and sufficient condition for term by term integration. In *Leçons II* [19] (p. 131) Lebesgue merely refers to the paper: « M. Vitali a écrit sur ce sujet un très important Mémoire, que je ne puis ici que signaler »; this is just before he gives DCT. In 1913, Camp, in a rather messy paper [2], gives a generalization of Vitali's theorem to several variables.

In 1915, de la Vallée Poussin, wrote a long paper [27] entitled « Sur l'intégrale de Lebesgue »; this article is complementary to his book *« Intégrales de Lebesgue, fonctions d'ensembles, classes de Baire »* [28] written at about the same time. In the paper, in the section on convergence theorems, de la Vallée Poussin discusses Vitali's work, and in the proof of Theorem 4 on p. 448–450 he simplifies considerably the hard part of Vitali's proof; we give a sketch of his argument.

It is clearly sufficient to prove that if $f_n \to 0$ on E, and $\int f_n$ are *not* uniformly absolutely continuous on E, then there exists $F \subseteq E$ such that $\int_F f_n \nrightarrow 0$. (We know this is true if $f_n \geq 0$ on E.) Let A_m be a sequence (to be chosen later) such that $0 < A_m < A_{m+1}$ for all $m \in \mathbf{N}$, and $A_m \to +\infty$, and let $E_m = \{x \in E : |f_n(x)| > A_m$ for some $n \in \mathbf{N}\}$. Note that the measure of E_m tends to zero.

Let $\varepsilon = \limsup \int_E |f_n|$; $\varepsilon > 0$, else $\int |f_n|$ are uniformly absolutely continuous. Let $\omega > 0$ with $\omega < \varepsilon/6$. It is fairly easy to choose A_m so that for each $m \in \mathbf{N}$, there exists $n \in \mathbf{N}$ such that the following three inequalities are satisfied:

$$(\ddagger) \qquad \int_{E \setminus E_m} |f_n| < \omega\,, \quad \int_{E_m} |f_n| > \varepsilon - \omega\,, \quad \int_{E_{m+1}} |f_n| < \omega\,.$$

This is done inductively: for each m, we can find n so that the first two inequalities are satisfied and then choose $A_{m+1} (> A_m)$ depending on n so that the third is satisfied. Further we can choose n increasing and $\to \infty$ with m.

($\ddagger$) implies that $\int_{E_m \setminus E_{m+1}} |f_n| > \varepsilon - 2\omega$, and so there exists $F_m \subseteq E_m \setminus E_{m+1}$ so that $\left| \int_{F_m} f_n \right| > \frac{1}{2}(\varepsilon - 2\omega)$. Put $F = \bigcup_{m=1}^{\infty} F_m$ (disjoint union). Note that

$F_i \subseteq E \backslash E_m$ for $i = 1, \dots, m-1$ and $F_h \subseteq E_{m+1}$ for $h = m+1, m+2, \dots$. Hence for each m, and the corresponding n,

$$\begin{aligned}\left|\int_F f_n\right| &\geq \left|\int_{F_m} f_n\right| - \sum_{i=1}^{m-1}\left|\int_{F_i} f_n\right| - \sum_{h=m+1}^{\infty}\left|\int_{F_h} f_n\right| \\ &\geq \left|\int_{F_m} f_n\right| - \int_{E\backslash E_m} |f_n| - \int_{E_{m+1}} |f_n| \\ &> \frac{\varepsilon - 2\omega}{2} - \omega - \omega = \frac{\varepsilon}{2} - 3\omega\,,\end{aligned}$$

which is positive since $\omega < \varepsilon/6$. Hence $\int_F f_n \not\to 0$, completing the proof.

The argument is very similar to Step 1 in Hahn's proof of the Vitali-Hahn-Saks Theorem [10] given in §3, and it is at least conceivable that Hahn got the initial impetus for his proof from de la Vallée Poussin's paper. In [27], Theorem 5 on p. 450, de la Vallée Poussin shows that uniform absolute continuity of $\int f_n$ on a space of finite non-atomic measure is equivalent to: *Given* $\varepsilon > 0$, *there exists* $K > 0$, *such that for all* $n \in \mathbf{N}$, $\int_{\{|f_n|>K\}} |f_n| < \varepsilon$.

This was rediscovered by Doob [3] 24 years later, the new criterion was called *uniform integrability*, and used extensively by Doob in his study of martingales. In 1918, H. Hahn [9, p. 1774] showed, using Vitali's result, that complete integrability implies strong L^1 convergence: this shows that Hahn was aware of Vitali's paper. However de la Vallée Poussin's paper seems to be virtually unknown! I learnt about it from the excellent set of bibliographical references on p. 223 of Hahn and Rosenthal [11]. Nagumo [21] discusses the theorem with reference to Vitali, uses it, and gives a necessary and sufficient condition for uniform absolute continuity. Vitali himself does not seem to have worked further on this subject. See the biographical article by A. Tonolo [26].

Among well-known books on real analysis written before World War II only Hobson [13] refers to Vitali's paper on p. 296–299. Hobson also has what is probably the first attempt to generalize Vitali's result to sets of infinite measure; these are all, in my opinion, somewhat artificial. Since the fifties some books on analysis and/or probability have included the concept of uniform absolute continuity or uniform integrability, but often without any mention of Vitali. Also, where there is a reference to Vitali, the result attributed to him is often the equivalence of uniform absolute continuity and strong L^1

convergence, which follows from the easy part of Vitali's work, whereas complete integrability is not mentioned. Rudin, *Real and Complex Analysis* [24], is an exception – in all three editions; however, in the first edition the Vitali convergence theorem is given, by the third edition this has changed to the Vitali-Hahn-Saks theorem. Dunford-Schwartz [6] has a comprehensive account in Chapters III and IV. Unfortunately there is a slip in the statement of Vitali's convergence theorem on p. 234.

3. The Vitali-Hahn-Saks Theorem

Vitali's convergence theorem is regarded as the origin of this theorem. It was first stated and proved by H. Hahn [10] in 1922. Hahn's statement and proof follow. (Both this result and Corollary 2 are referred to as "The Vitali-Hahn-Saks Theorem". The result is obviously stronger than Vitali's convergence theorem.)

THEOREM (H. Hahn [10] Thm. XXI, pages 45–50). *If $m(E) < \infty$, f_n integrable on E, and for each measurable $F \subseteq E$, $\lim_{n\to\infty} \int_F f_n$ exists and is finite, then $\int f_n$ are uniformly absolutely continuous.*

Proof. Again, all the sets that occur in this proof will be measurable. Suppose the integrals are not uniformly absolutely continuous. Then there exists $\varepsilon > 0$ with the property that for each $N \in \mathbf{N}$ and $\sigma > 0$ there is a measurable set Z with $m(Z) < \sigma$ and $n_0 > N$ with $\int_Z |f_{n_0}| > \varepsilon$. By considering the sets where $f_{n_0} \geq 0$ and $f_{n_0} \leq 0$, we obtain for each $N \in \mathbf{N}$, a set M with $m(M) < \sigma$ and $n_0 > N$ with $\left|\int_M f_{n_0}\right| > \frac{\varepsilon}{2}$.

STEP 1. We show that there exists a sequence of pairwise disjoint sets M_ν and an increasing sequence of positive integers n_ν such that

$$\left|\int_{M_\nu} f_{n_\nu}\right| \geq \frac{\varepsilon}{2} \text{ for all } \nu \in \mathbf{N}.$$

We start by choosing a proper subset Z_1 of E and $n_1 \in \mathbf{N}$ such that $\left|\int_{Z_1} f_{n_1}\right| > \frac{\varepsilon}{2}$. We observe that there exists $\sigma > 0$ sufficiently small so that if $Z' \subset Z_1$ with $m(Z') < \sigma$, then we still have $\left|\int_{Z_1 \setminus Z'} f_{n_1}\right| > \frac{\varepsilon}{2}$. Now by our assumption we obtain $n_2 > n_1$ and a set Z_2 with $m(Z_2) < \sigma$ but $\left|\int_{Z_2} f_{n_2}\right| > \frac{\varepsilon}{2}$. Since $m(Z_1 \cap Z_2) < \sigma$, we have $\left|\int_{Z_1 \setminus Z_1 \cap Z_2} f_{n_1}\right| > \frac{\varepsilon}{2}$ and $\left|\int_{Z_2} f_{n_2}\right| > \frac{\varepsilon}{2}$.

In the same way we obtain $n_3 > n_2$ and a set Z_3 of sufficiently small measure so that

$$\left|\int_{Z_1 \setminus Z_1 \cap (Z_2 \cup Z_3)} f_{n_1}\right| > \frac{\varepsilon}{2}; \quad \left|\int_{Z_2 \setminus (Z_2 \cap Z_3)} f_{n_2}\right| > \frac{\varepsilon}{2}; \quad \left|\int_{Z_3} f_{n_3}\right| > \frac{\varepsilon}{2}.$$

We continue in this way to obtain a strictly increasing sequence n_ν and sets Z_ν, so that

$$\left|\int_{Z_1 \setminus Z_1 \cap (Z_2 \cup \cdots \cup Z_\nu)} f_{n_1}\right| > \frac{\varepsilon}{2}; \quad \left|\int_{Z_2 \setminus Z_2 \cap (Z_3 \cup \cdots \cup Z_\nu)} f_{n_2}\right| > \frac{\varepsilon}{2}; \ldots ; \left|\int_{Z_\nu} f_{n_\nu}\right| > \frac{\varepsilon}{2}.$$

Set

$$M_1 = Z_1 \setminus Z_1 \cap \bigcup_{j=2}^{\infty} Z_j$$

$$M_2 = Z_2 \setminus Z_2 \cap \bigcup_{j=3}^{\infty} Z_j$$

$$\vdots \quad \vdots$$

$$M_\nu = Z_\nu \setminus Z_\nu \cap \bigcup_{j=\nu+1}^{\infty} Z_j .$$

The M_ν are pairwise disjoint and $\left|\int_{M_\nu} f_{n_\nu}\right| \geq \frac{\varepsilon}{2}$ for all ν, completing the proof of Step 1.

STEP 2. We know that for each measurable set M, $\lim_{n\to\infty} \int_M f_n$ exists and is finite; we shall show using Step 1, that $\lim_{n\to\infty} \int_M f_n$ does not exist for some M – this will complete the proof of the Theorem. Let M_1 and n_1 be as in Step 1, we put $G_1 = M_1$ and $\nu_1 = 1$ so that $n_{\nu_1} = n_1$. By absolute continuity of $\int f_{n_1}$, there exists $\rho_1 > 0$ such that if $m(Z) < \rho_1$, then $\left|\int_Z f_{n_1}\right| < \frac{\varepsilon}{12}$. Since $\lim_{n\to\infty} \int_{G_1} f_n$ exists (and is finite), there exists $N_1 > n_1$ such that

$$\left|\int_{G_1} f_n - \int_{G_1} f_{n'}\right| < \frac{\varepsilon}{12} \quad \text{if } n \geq N_1,\ n' \geq N_1\,.$$

Now there exists σ_1, $0 < \sigma_1 < \rho_1$, such that if $m(Z) < \sigma_1$, then $\left|\int_Z f_{N_1}\right| < \frac{\varepsilon}{12}$. Since the sets M_ν are disjoint measurable subsets of E, and $m(E) < \infty$, there exists ν_2 such that $n_{\nu_2} > N_1$ and $\sum_{\nu=\nu_2}^{\infty} m(M_\nu) < \sigma_1$. Put $G_2 = M_{\nu_1} \cup M_{\nu_2}$.

Now there exists ρ_2, $0 < \rho_2 < \sigma_1$, such that if $m(Z) < \rho_2$ then $\left|\int_Z f_{n_{\nu_2}}\right| < \frac{\varepsilon}{12}$. Since $\lim_{n\to\infty} \int_{G_2} f_n$ exists, there exists $N_2 > n_{\nu_2}$ so that

$$\left|\int_{G_2} f_n - \int_{G_2} f_{n'}\right| < \frac{\varepsilon}{12} \quad \text{if } n \geq N_2,\ n' \geq N_2\,.$$

Again there exists σ_2, $0 < \sigma_2 < \rho_2$ such that if $m(Z) < \sigma_2$, then $\left|\int_Z f_{N_2}\right| < \frac{\varepsilon}{12}$; and there exists ν_3, with $n_{\nu_3} > N_2$ and $\sum_{\nu=\nu_3}^{\infty} m(M_\nu) < \sigma_2$. Put $G_3 = M_{\nu_1} \cup M_{\nu_2} \cup M_{\nu_3}$. Proceeding in this way, we get:

Two sequences of positive integers n_{ν_j} and N_j with

(I) $$n_{\nu_1} < N_1 < n_{\nu_2} < N_2 < \cdots < n_{\nu_i} < N_i < \cdots\,.$$

Two sequences of positive numbers ρ_j and σ_j with

(II) $$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots > \rho_i > \sigma_i > \cdots\,.$$

These (together with the M_ν and f_n) have the properties:

(III) $$\sum_{\nu=\nu_{i+1}}^{\infty} m(M_\nu) < \sigma_i < \rho_i\,,$$

where M_ν are as in Step 1;

(IV) $$\left|\int_Z f_{n_{\nu_i}}\right| < \frac{\varepsilon}{12} \quad \text{if } m(Z) < \rho_i\,,$$

(V) $$\left|\int_Z f_{N_i}\right| < \frac{\varepsilon}{12} \quad \text{if } m(Z) < \sigma_i\,.$$

Further, if $G_i = \bigcup_{j=1}^{i} M_{\nu_j}$, then

(VI) $$\left|\int_{G_i} f_n - \int_{G_i} f_{n'}\right| < \frac{\varepsilon}{12} \quad \text{if } n \geq N_i,\ n' \geq N_i\,.$$

Recall from Step 1 that

$$\text{(VII)} \qquad \left|\int_{M_{\nu_i}} f_{n_{\nu_i}}\right| \geq \frac{\varepsilon}{2} \quad \text{for all } \nu_i\,.$$

Now put $M = \bigcup_{i=1}^{\infty} M_{\nu_i}$ (recall these sets are disjoint) and $R_i = M \backslash G_i$. Now

$$\begin{aligned}\text{(VIII)} \quad & \left|\int_M f_{N_{i-1}} - \int_M f_{n_{\nu_i}}\right| \\ &= \left|\int_{G_{i-1}} f_{N_{i-1}} - \int_{G_{i-1}} f_{n_{\nu_i}} + \int_{R_{i-1}} f_{N_{i-1}} - \int_{M_{\nu_i}} f_{n_{\nu_i}} - \int_{R_i} f_{n_{\nu_i}}\right| \\ &\geq \left|\int_{M_{\nu_i}} f_{n_{\nu_i}}\right| - \left|\int_{G_{i-1}} f_{N_{i-1}} - \int_{G_{i-1}} f_{n_{\nu_i}}\right| - \left|\int_{R_{i-1}} f_{N_{i-1}}\right| - \left|\int_{R_i} f_{n_{\nu_i}}\right|.\end{aligned}$$

By (VI), $\left|\int_{G_{i-1}} f_{N_{i-1}} - \int_{G_{i-1}} f_{n_{\nu_i}}\right| < \frac{\varepsilon}{12}$. Since $m(R_{i-1}) < \sigma_{i-1}$, by (V) $\left|\int_{R_{i-1}} f_{N_{i-1}}\right| < \frac{\varepsilon}{12}$; and since $m(R_i) < \rho_i$, by (IV) $\left|\int_{R_i} f_{n_{\nu_i}}\right| < \frac{\varepsilon}{12}$. These assertions together with (VII) show that (VIII) implies that

$$\left|\int_M f_{N_{i-1}} - \int_M f_{n_{\nu_i}}\right| \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{12} - \frac{\varepsilon}{12} - \frac{\varepsilon}{12} = \frac{\varepsilon}{4} \quad \text{for all } i \in \mathbf{N}\,.$$

Hence $\int_M f_n$ cannot have a finite limit as $n \to \infty$, contradicting our hypothesis and completing the proof of (Step 2 and) the theorem.

REMARK. It is easy to prove also that $\int_E f_n$ are uniformly bounded.

COROLLARY 1. *Under the same hypothesis as in the theorem the set function* $\nu(F) = \lim_{n\to\infty} \int_F f_n$ *is absolutely continuous with respect to* m *and so is countably additive and is the integral of an integrable function* f.

COROLLARY 2 (Vitali-Hahn-Saks Theorem). *If* ν_n *is a sequence of finite countably additive set functions on a* σ*-algebra* $\mathbf{M}$ *of subsets of* E *and* $\lim_{n\to\infty} \nu_n(F)$ *exists and is finite for all* $F \in \mathbf{M}$, *then this limit is countably additive on* $(E, \mathbf{M})$.

Proof. Put $m(F) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\nu_n|(F)}{|\nu_n|(E)}$, where $|\nu_n|$ is the total variation of ν_n. Then m is a finite measure on E and each ν_n is absolutely continuous with respect to m. By the Lebesgue-Radon-Nikodym Theorem $\nu_n = \int f_n dm$ for some integrable f_n. Apply the theorem and Corollary 1.

NOTE. As remarked earlier, the above is the first proof of the Vitali-Hahn-Saks Theorem. Note that no use is made in the proof of the Baire Category Theorem. The proof is somewhat similar to Vitali's proof of his convergence theorem, and even more so to de la Vallée Poussin's proof, although no reference is made to Vitali or de la Vallée Poussin. Hahn does refer however to the paper of B.H. Camp [2], which generalizes Vitali's result to several variables; Camp does mention Vitali. Corollary 2 was first stated and proved by Nikodym [22], [23], ten years after Hahn. The proof is direct, but Nikodym remarks that it is analogous to Hahn's theorem for integrals; he also says that his results are « en solidarité étroite avec des théorèmes de M. H. Hahn ». He does not remark that his result is an immediate corollary of Hahn's. This is somewhat surprising, since he only needs to use a famous result that he had just proved: the Lebesgue-Radon-Nikodym Theorem.

Hahn's proof precedes by ten years the Baire category proof of Saks [25], who also gives Corollary 2. This proof was apparently also discovered independently by Banach, and is included in the Polish edition of his book on linear operators, *Teorja Operacyj* [1], but not in the much better known French edition. Saks mentions Hahn's result, but not Vitali's. This proof is given in many books. A very detailed account (including generalizations to vector-valued measures) of the theorems of Vitali and Vitali-Hahn-Saks is given in Chapters III and IV of Dunford-Schwartz [6], specifically on pages 122, 150 and 234 of Chapter III and pages 292–295, 306 and 389 of Chapter IV. Page 389 also gives references to some interesting generalizations especially of Corollary 2 by Dubrovsky and Cafiero. A proof of Corollary 2 above was given in 1945 by Y. Dubrovsky [5]; it is somewhat similar to the proofs of Hahn and Nikodym. This proof was simplified by Y. N. Dowker [4]. Her argument is given on pages 32–35 of N. Friedman's little book on Ergodic Theory [8]. No one mentions Hahn, though Dowker mentions Saks.

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[1]) Lebesgue's collected works have been published as: LEBESGUE, Henri Léon. *Œuvres Scientifiques.* 5 volumes. *L'Enseignement Mathématique*, Institut de Mathématiques, Univ. de Genève, 1972. Vol. 1 contains *'thèse'*; vol. 2 contains *Leçons I* and Lebesgue (1910); vol. 3 contains Lebesgue (1908) and (1909). *Leçons II* is *not* included.

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