3. The Vitali-Hahn-Saks Theorem

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convergence, which follows from the easy part of Vitali's work, whereas complete integrability is not mentioned. Rudin, *Real and Complex Analysis* [24], is an exception – in all three editions; however, in the first edition the Vitali convergence theorem is given, by the third edition this has changed to the Vitali-Hahn-Saks theorem. Dunford-Schwartz [6] has a comprehensive account in Chapters III and IV. Unfortunately there is a slip in the statement of Vitali's convergence theorem on p. 234.

3. The Vitali-Hahn-Saks Theorem

Vitali's convergence theorem is regarded as the origin of this theorem. It was first stated and proved by H. Hahn [10] in 1922. Hahn's statement and proof follow. (Both this result and Corollary 2 are referred to as "The Vitali-Hahn-Saks Theorem". The result is obviously stronger than Vitali's convergence theorem.)

THEOREM (H. Hahn [10] Thm. XXI, pages 45–50). *If* $m(E) < \infty$, f_n *integrable on* E, *and for each measurable* $F \subseteq E$, $\lim_{n\to\infty} \int_F f_n$ *exists and is finite, then* $\int f_n$ *are uniformly absolutely continuous.*

Proof. Again, all the sets that occur in this proof will be measurable. Suppose the integrals are not uniformly absolutely continuous. Then there exists $\varepsilon > 0$ with the property that for each $N \in \mathbf{N}$ and $\sigma > 0$ there is a measurable set Z with $m(Z) < \sigma$ and $n_0 > N$ with $\int_Z |f_{n_0}| > \varepsilon$. By considering the sets where $f_{n_0} \geq 0$ and $f_{n_0} \leq 0$, we obtain for each $N \in \mathbf{N}$, a set M with $m(M) < \sigma$ and $n_0 > N$ with $\left|\int_M f_{n_0}\right| > \frac{\varepsilon}{2}$.

STEP 1. We show that there exists a sequence of pairwise disjoint sets M_ν and an increasing sequence of positive integers n_ν such that

$$\left|\int_{M_\nu} f_{n_\nu}\right| \geq \frac{\varepsilon}{2} \text{ for all } \nu \in \mathbf{N}.$$

We start by choosing a proper subset Z_1 of E and $n_1 \in \mathbf{N}$ such that $\left|\int_{Z_1} f_{n_1}\right| > \frac{\varepsilon}{2}$. We observe that there exists $\sigma > 0$ sufficiently small so that if $Z' \subset Z_1$ with $m(Z') < \sigma$, then we still have $\left|\int_{Z_1\setminus Z'} f_{n_1}\right| > \frac{\varepsilon}{2}$. Now by our assumption we obtain $n_2 > n_1$ and a set Z_2 with $m(Z_2) < \sigma$ but $\left|\int_{Z_2} f_{n_2}\right| > \frac{\varepsilon}{2}$. Since $m(Z_1 \cap Z_2) < \sigma$, we have $\left|\int_{Z_1\setminus Z_1\cap Z_2} f_{n_1}\right| > \frac{\varepsilon}{2}$ and $\left|\int_{Z_2} f_{n_2}\right| > \frac{\varepsilon}{2}$.

In the same way we obtain $n_3 > n_2$ and a set Z_3 of sufficiently small measure so that

$$\left|\int_{Z_1\setminus Z_1\cap(Z_2\cup Z_3)} f_{n_1}\right| > \frac{\varepsilon}{2}; \quad \left|\int_{Z_2\setminus(Z_2\cap Z_3)} f_{n_2}\right| > \frac{\varepsilon}{2}; \quad \left|\int_{Z_3} f_{n_3}\right| > \frac{\varepsilon}{2}.$$

We continue in this way to obtain a strictly increasing sequence n_ν and sets Z_ν, so that

$$\left|\int_{Z_1\setminus Z_1\cap(Z_2\cup\cdots\cup Z_\nu)} f_{n_1}\right| > \frac{\varepsilon}{2}; \quad \left|\int_{Z_2\setminus Z_2\cap(Z_3\cup\cdots\cup Z_\nu)} f_{n_2}\right| > \frac{\varepsilon}{2}; \ldots; \left|\int_{Z_\nu} f_{n_\nu}\right| > \frac{\varepsilon}{2}.$$

Set

$$M_1 = Z_1 \setminus Z_1 \cap \bigcup_{j=2}^{\infty} Z_j$$

$$M_2 = Z_2 \setminus Z_2 \cap \bigcup_{j=3}^{\infty} Z_j$$

$$\vdots \quad \vdots$$

$$M_\nu = Z_\nu \setminus Z_\nu \cap \bigcup_{j=\nu+1}^{\infty} Z_j .$$

The M_ν are pairwise disjoint and $\left|\int_{M_\nu} f_{n_\nu}\right| \geq \frac{\varepsilon}{2}$ for all ν, completing the proof of Step 1.

STEP 2. We know that for each measurable set M, $\lim_{n\to\infty} \int_M f_n$ exists and is finite; we shall show using Step 1, that $\lim_{n\to\infty} \int_M f_n$ does not exist for some M – this will complete the proof of the Theorem. Let M_1 and n_1 be as in Step 1, we put $G_1 = M_1$ and $\nu_1 = 1$ so that $n_{\nu_1} = n_1$. By absolute continuity of $\int f_{n_1}$, there exists $\rho_1 > 0$ such that if $m(Z) < \rho_1$, then $\left|\int_Z f_{n_1}\right| < \frac{\varepsilon}{12}$. Since $\lim_{n\to\infty} \int_{G_1} f_n$ exists (and is finite), there exists $N_1 > n_1$ such that

$$\left|\int_{G_1} f_n - \int_{G_1} f_{n'}\right| < \frac{\varepsilon}{12} \quad \text{if } n \geq N_1,\ n' \geq N_1 .$$

Now there exists σ_1, $0 < \sigma_1 < \rho_1$, such that if $m(Z) < \sigma_1$, then $\left|\int_Z f_{N_1}\right| < \frac{\varepsilon}{12}$. Since the sets M_ν are disjoint measurable subsets of E, and $m(E) < \infty$, there exists ν_2 such that $n_{\nu_2} > N_1$ and $\sum_{\nu=\nu_2}^{\infty} m(M_\nu) < \sigma_1$. Put $G_2 = M_{\nu_1} \cup M_{\nu_2}$.

Now there exists ρ_2, $0 < \rho_2 < \sigma_1$, such that if $m(Z) < \rho_2$ then $\left|\int_Z f_{n_{\nu_2}}\right| < \frac{\varepsilon}{12}$. Since $\lim_{n\to\infty} \int_{G_2} f_n$ exists, there exists $N_2 > n_{\nu_2}$ so that

$$\left|\int_{G_2} f_n - \int_{G_2} f_{n'}\right| < \frac{\varepsilon}{12} \quad \text{if } n \geq N_2,\ n' \geq N_2 .$$

Again there exists σ_2, $0 < \sigma_2 < \rho_2$ such that if $m(Z) < \sigma_2$, then $\left|\int_Z f_{N_2}\right| < \frac{\varepsilon}{12}$; and there exists ν_3, with $n_{\nu_3} > N_2$ and $\sum_{\nu=\nu_3}^{\infty} m(M_\nu) < \sigma_2$. Put $G_3 = M_{\nu_1} \cup M_{\nu_2} \cup M_{\nu_3}$. Proceeding in this way, we get:

Two sequences of positive integers n_{ν_j} and N_j with

(I) $$n_{\nu_1} < N_1 < n_{\nu_2} < N_2 < \cdots < n_{\nu_i} < N_i < \cdots .$$

Two sequences of positive numbers ρ_j and σ_j with

(II) $$\rho_1 > \sigma_1 > \rho_2 > \sigma_2 > \cdots > \rho_i > \sigma_i > \cdots .$$

These (together with the M_ν and f_n) have the properties:

(III) $$\sum_{\nu=\nu_{i+1}}^{\infty} m(M_\nu) < \sigma_i < \rho_i ,$$

where M_ν are as in Step 1;

(IV) $$\left|\int_Z f_{n_{\nu_i}}\right| < \frac{\varepsilon}{12} \qquad \text{if } m(Z) < \rho_i ,$$

(V) $$\left|\int_Z f_{N_i}\right| < \frac{\varepsilon}{12} \qquad \text{if } m(Z) < \sigma_i .$$

Further, if $G_i = \bigcup_{j=1}^{i} M_{\nu_j}$, then

(VI) $$\left|\int_{G_i} f_n - \int_{G_i} f_{n'}\right| < \frac{\varepsilon}{12} \qquad \text{if } n \geq N_i,\ n' \geq N_i .$$

Recall from Step 1 that

$$\text{(VII)} \qquad \left|\int_{M_{\nu_i}} f_{n_{\nu_i}}\right| \geq \frac{\varepsilon}{2} \quad \text{for all } \nu_i\,.$$

Now put $M = \bigcup_{i=1}^{\infty} M_{\nu_i}$ (recall these sets are disjoint) and $R_i = M \backslash G_i$. Now

$$\begin{aligned}
\text{(VIII)} \quad & \left|\int_M f_{N_{i-1}} - \int_M f_{n_{\nu_i}}\right| \\
&= \left|\int_{G_{i-1}} f_{N_{i-1}} - \int_{G_{i-1}} f_{n_{\nu_i}} + \int_{R_{i-1}} f_{N_{i-1}} - \int_{M_{\nu_i}} f_{n_{\nu_i}} - \int_{R_i} f_{n_{\nu_i}}\right| \\
&\geq \left|\int_{M_{\nu_i}} f_{n_{\nu_i}}\right| - \left|\int_{G_{i-1}} f_{N_{i-1}} - \int_{G_{i-1}} f_{n_{\nu_i}}\right| - \left|\int_{R_{i-1}} f_{N_{i-1}}\right| - \left|\int_{R_i} f_{n_{\nu_i}}\right|.
\end{aligned}$$

By (VI), $\left|\int_{G_{i-1}} f_{N_{i-1}} - \int_{G_{i-1}} f_{n_{\nu_i}}\right| < \frac{\varepsilon}{12}$. Since $m(R_{i-1}) < \sigma_{i-1}$, by (V) $\left|\int_{R_{i-1}} f_{N_{i-1}}\right| < \frac{\varepsilon}{12}$; and since $m(R_i) < \rho_i$, by (IV) $\left|\int_{R_i} f_{n_{\nu_i}}\right| < \frac{\varepsilon}{12}$. These assertions together with (VII) show that (VIII) implies that

$$\left|\int_M f_{N_{i-1}} - \int_M f_{n_{\nu_i}}\right| \geq \frac{\varepsilon}{2} - \frac{\varepsilon}{12} - \frac{\varepsilon}{12} - \frac{\varepsilon}{12} = \frac{\varepsilon}{4} \quad \text{for all } i \in \mathbf{N}\,.$$

Hence $\int_M f_n$ cannot have a finite limit as $n \to \infty$, contradicting our hypothesis and completing the proof of (Step 2 and) the theorem.

REMARK. It is easy to prove also that $\int_E f_n$ are uniformly bounded.

COROLLARY 1. *Under the same hypothesis as in the theorem the set function* $\nu(F) = \lim_{n\to\infty} \int_F f_n$ *is absolutely continuous with respect to* m *and so is countably additive and is the integral of an integrable function* f.

COROLLARY 2 (Vitali-Hahn-Saks Theorem). *If* ν_n *is a sequence of finite countably additive set functions on a* σ*-algebra* $\mathbf{M}$ *of subsets of* E *and* $\lim_{n\to\infty} \nu_n(F)$ *exists and is finite for all* $F \in \mathbf{M}$, *then this limit is countably additive on* $(E, \mathbf{M})$.

Proof. Put $m(F) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\nu_n|(F)}{|\nu_n|(E)}$, where $|\nu_n|$ is the total variation of ν_n. Then m is a finite measure on E and each ν_n is absolutely continuous with respect to m. By the Lebesgue-Radon-Nikodym Theorem $\nu_n = \int f_n dm$ for some integrable f_n. Apply the theorem and Corollary 1.

NOTE. As remarked earlier, the above is the first proof of the Vitali-Hahn-Saks Theorem. Note that no use is made in the proof of the Baire Category Theorem. The proof is somewhat similar to Vitali's proof of his convergence theorem, and even more so to de la Vallée Poussin's proof, although no reference is made to Vitali or de la Vallée Poussin. Hahn does refer however to the paper of B.H. Camp [2], which generalizes Vitali's result to several variables; Camp does mention Vitali. Corollary 2 was first stated and proved by Nikodym [22], [23], ten years after Hahn. The proof is direct, but Nikodym remarks that it is analogous to Hahn's theorem for integrals; he also says that his results are « en solidarité étroite avec des théorèmes de M. H. Hahn ». He does not remark that his result is an immediate corollary of Hahn's. This is somewhat surprising, since he only needs to use a famous result that he had just proved: the Lebesgue-Radon-Nikodym Theorem.

Hahn's proof precedes by ten years the Baire category proof of Saks [25], who also gives Corollary 2. This proof was apparently also discovered independently by Banach, and is included in the Polish edition of his book on linear operators, *Teorja Operacyj* [1], but not in the much better known French edition. Saks mentions Hahn's result, but not Vitali's. This proof is given in many books. A very detailed account (including generalizations to vector-valued measures) of the theorems of Vitali and Vitali-Hahn-Saks is given in Chapters III and IV of Dunford-Schwartz [6], specifically on pages 122, 150 and 234 of Chapter III and pages 292–295, 306 and 389 of Chapter IV. Page 389 also gives references to some interesting generalizations especially of Corollary 2 by Dubrovsky and Cafiero. A proof of Corollary 2 above was given in 1945 by Y. Dubrovsky [5]; it is somewhat similar to the proofs of Hahn and Nikodym. This proof was simplified by Y. N. Dowker [4]. Her argument is given on pages 32–35 of N. Friedman's little book on Ergodic Theory [8]. No one mentions Hahn, though Dowker mentions Saks.

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