4.1 Proof of Theorems 2.2, 2.6 and 2.10

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where the dots indicate a linear combination of the remaining vectors. It follows from (3.6) that

$$\Delta_{n-d+i} = (-1)^{i-1}\, b\, \Delta'_{n-d+i}, \quad \Delta_{n-d+i+1} = (-1)^{d-i}\, a\, \Delta'_{n-d+i}. \tag{3.8}$$

It is time to use the strict convexity of P. Let H be a hyperplane in $\mathbf{RP}^d$ through $d-1$ vertices $V_{n-d+i+1}, \ldots, \widehat{V}_{n+1}, \ldots, V_{n+i}$ which intersects P with multiplicity $d-1$, and let $\widetilde{H}$ be its lifting to $\mathbf{R}^{d+1}$. Choose a linear function φ in $\mathbf{R}^{d+1}$ vanishing on $\widetilde{H}$ and such that $\varphi(\widetilde{V}_{n+1}) > 0$. We claim that

$$(-1)^{d-i}\varphi(\widetilde{V}_{n-d+i}) > 0 \quad \text{and} \quad (-1)^{i-1}\varphi(\widetilde{V}_n) > 0. \tag{3.9}$$

Indeed, by Lemma 3.3, the intersection multiplicities of $\widetilde{H}$ with the polygonal lines $(\widetilde{V}_{n-d+i}, \ldots, \widetilde{V}_{n+1})$ and $(\widetilde{V}_{n+1}, \ldots, \widetilde{V}_{n+i+1})$ are at least $d-i$ and $i-1$, respectively. Since H intersects P with multiplicity $d-1$, the above two multiplicities are indeed equal to $d-i$ and $i-1$. The inequalities (3.9) now readily follow from Lemma 3.5.

Finally, we evaluate φ on (3.7):

$$\varphi(\widetilde{V}_{n+1}) = a\,\varphi(\widetilde{V}_{n-d+i}) + b\,\varphi(\widetilde{V}_{n+i+1}).$$

It follows from (3.9) and the inequality $\varphi(\widetilde{V}_{n+1}) > 0$ that at least one of the numbers $(-1)^{i-1}b$ and $(-1)^{d-i}a$ is positive. In view of (3.8), Lemma 3.12 follows. □

Thus Theorem 3.11 is also proved. □

REMARK 3.13. Strict convexity is necessary for the existence of $d+1$ flattenings. One can easily construct a closed polygon without any flattenings and even C^0-approximate an arbitrary closed smooth curve by such polygons. In the smooth case such an approximation is well known: given a curve γ_0, the approximating one, γ, spirals around in a tubular neighbourhood of γ_0. In the polygonal case we take a sufficiently fine straightening of γ.

4. APPLICATIONS OF THE MAIN THEOREM

4.1 PROOF OF THEOREMS 2.2, 2.6 AND 2.10

Now we prove the results announced in Section 2. The idea is the same in all three cases and is precisely that of Barner's proof of the smooth versions of these theorems – see [3] and also [15]. We will consider Theorem 2.6 in detail, indicating the necessary changes in the other two cases.

Let P be as in Theorem 2.6. We consider the Veronese mapping $\mathcal{V}\colon \mathbf{RP}^2 \to \mathbf{RP}^5$ of degree 2:

$$\mathcal{V}\colon (x:y:z) \mapsto (x^2 : y^2 : z^2 : xy : yz : zx)\,. \tag{4.1}$$

The image $\mathcal{V}(P)$ is a piecewise smooth curve. Every edge is homotopic to a straight segment, with the endpoints $\mathcal{V}(V_i), \mathcal{V}(V_{i+1})$ fixed, and we obtain a polygon Q in $\mathbf{RP}^5$. Assume first that Q is in general position.

LEMMA 4.1. *A quintuple $\mathcal{V}(V_i), \ldots, \mathcal{V}(V_{i+4})$ is a flattening of Q if and only if $(V_i, \ldots, V_{i+4})$ is an extremal quintuple of vertices of P.*

Proof. The Veronese map establishes a one-to-one correspondence between conics in $\mathbf{RP}^2$ and hyperplanes in $\mathbf{RP}^5$: the image of a conic is the intersection of a hyperplane with the quadric surface $\mathcal{V}(\mathbf{RP}^2)$. Since $\mathcal{V}$ is an embedding, the points V_{i-1} and V_{i+5} lie on one side of the conic through $(V_i, \ldots, V_{i+4})$ if and only if the points $\mathcal{V}(V_{i-1})$ and $\mathcal{V}(V_{i+5})$ lie on one side of the corresponding hyperplane. □

Next we show that the polygon Q is strictly convex. Given 4 indices i_1, i_2, i_3, i_4, we consider two lines in $\mathbf{RP}^2$: (V_{i_1}, V_{i_2}) and (V_{i_3}, V_{i_4}) ; the union of these lines is a conic that does not meet P any more. The corresponding hyperplane in $\mathbf{RP}^5$ contains the vertices $\mathcal{V}(V_{i_1}), \mathcal{V}(V_{i_2}), \mathcal{V}(V_{i_3}), \mathcal{V}(V_{i_4})$ and intersects Q with multiplicity 4.

Theorem 2.6 now follows from Theorem 3.11 for $d = 5$, provided Q is in general position. Otherwise, we replace P by a convex polygon P', close to P, such that the corresponding polygon Q' is in general position. Then, as above, P' has at least 6 extremal quintuples of vertices, and therefore so does P. This completes the proof.

To prove Theorems 2.2 and 2.10, one replaces the map (4.1) by the Veronese map $\mathcal{V} : \mathbf{RP}^2 \to \mathbf{RP}^3$

$$\mathcal{V} : (x:y:z) \mapsto (x^2 + y^2 : z^2 : yz : zx)$$

and by the Segre map $\mathcal{S}\colon \mathbf{RP}^1 \times \mathbf{RP}^1 \to \mathbf{RP}^3$

$$\mathcal{S}\colon ((x_1 : y_1), (x_2 : y_2)) \mapsto (x_1x_2 : x_1y_2 : y_1x_2 : y_1y_2)\,,$$

respectively. The proofs of strict convexity for the corresponding polygons Q reproduce those in the smooth case (see [15]).