2. CONJUGATION AND PARTIAL CONJUGATION

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well as the work of Mellor and Thurston, of course) shows the existence of non-trivial finite type link concordance invariants.

To extend the applicability of our general philosophy slightly, we find that the operation on the vector $\{\mu(rst)\}$ induced by reversing the orientation of each component of a string link is to change it by a negative sign followed by a translation whose translation vector's coordinates are quadratic polynomials in l_{ij}. If the dimension of the subspace generated by this vector together with the translation vectors of conjugations and partial conjugations is still less than $\binom{k}{3}$ for generic values of the linking numbers, and this is the case indeed, we can construct a non-trivial link-homotopy invariant polynomial which is changed by a sign when the orientation of each component of a link is reversed. We say that such a link invariant detects the invertibility for links. Recall that the reversion of the orientation of every component of a link does not change the quantum invariant associated with an irreducible representation of a semi-simple Lie algebra (see, for example, [8]). Thus our invariant is of finite type but is not determined by quantum invariants. The existence of a finite type knot invariant which detects the invertibility for knots is a major problem in the theory of finite type invariants (see, for example, [8] and [4]). We believe that finite type knot invariants can not detect the invertibility for knots.

It remains unclear whether we can have a complete set of link-homotopy invariant polynomials which determines uniquely link-homotopy classes of links. See [5] for an earlier attempt on this problem[2]). This problem could probably be translated to the problem of understanding the sublattice generated by the translation vectors of conjugations and partial conjugations. A better understanding of this sublattice might also be useful in answering the following question. If we let $\deg(l_{ij}) = 1$ and $\deg(\mu(rst)) = 2$, the link-homotopy invariant polynomial for $k = 6$ we construct in Section 3, which detects the invertibility for links, is a linear combination of 113,700 monomials of degree 22, homogeneous in both l_{ij} and $\mu(rst)$ and linear in $\mu(rst)$. Is there a shorter link-homotopy invariant polynomial detecting the invertibility for links ?

2. Conjugation and partial conjugation

We first recall the classification of ordered, oriented links up to link-homotopy given in [3]. We will follow the notations of [3].

[2]) See [6] for another approach to the similar problem for surgery equivalence of links. Notice that both approaches attempted to reduce the indeterminacies of the $\bar{\mu}$-invariants.

Let $\mathcal{H}(k)$ be the group of link-homotopy classes of ordered, oriented string links with k components. The components of a string link will be ordered by $1, 2, \ldots, k$. Recall that a string link is a concordance of k marked points inside of the 2-disk D^2 to itself in $D^2 \times [0,1]$, such that it has no closed component. Two string links are link-homotopic if they are homotopic in such a way that at any moment of the homotopy, different components remain disjoint (but they are allowed to have self-intersections). Two string links can be put together to form a new string link and this gives rise to a group structure on the set of all link-homotopy classes of string links. This is the group $\mathcal{H}(k)$.

A pure braid is by definition a string link of the same number of components. So we have a natural map from the pure braid group $P(k)$ of k components to $\mathcal{H}(k)$. It is shown in [3] that this natural map $P(k) \to \mathcal{H}(k)$ is onto.

Deletion of the i^{th} component of the string link gives rise to a group homomorphism $d_i \colon \mathcal{H}(k) \to \mathcal{H}(k-1)$. If $F(k)$ denotes the free group of rank k generated by $x_1, x_2, \ldots, x_k$, the reduced free group $RF(k)$ is the quotient of $F(k)$ by adding relations $[x_i, x_i^g] = 1$ for all i and all $g \in F(k)$.

LEMMA 2.1. *There is a split short exact sequence of groups*

$$1 \longrightarrow RF(k-1) \longrightarrow \mathcal{H}(k) \xrightarrow{d_i} \mathcal{H}(k-1) \longrightarrow 1 \tag{1}$$

where $RF(k-1)$ is the reduced free group generated by $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k$.

Notice that the split exact sequence (1) depends on the deleted component so that there are k such split exact sequences altogether. A split exact sequence determines a semi-direct product decomposition

$$\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1).$$

Conjugation in the group $\mathcal{H}(k)$ is defined as usual: A conjugation of $\sigma \in \mathcal{H}(k)$ by $\beta \in \mathcal{H}(k)$ is the element $\beta\sigma\beta^{-1} \in \mathcal{H}(k)$. A *partial conjugation* of $\sigma \in \mathcal{H}(k)$ is an element of the form θhgh^{-1}, where we write $\sigma = \theta g$ according to a decomposition $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$, for $\theta \in \mathcal{H}(k-1)$ and $g \in RF(k-1)$, and for an arbitrary $h \in RF(k-1)$.

To form the closure of a string link $\sigma \in \mathcal{H}(k)$, we may think of it as a pure braid in $P(k)$ and its closure will be the usual braid closure. The closure of $\sigma \in \mathcal{H}(k)$ is an ordered, oriented link of k components. It is not hard to see that every link-homotopy class of ordered, oriented links with k components can be realized as the closure of an element in $\mathcal{H}(k)$, and thus the closure of a pure braid in $P(k)$. One of the main results of [3] is the following classification theorem.

THEOREM 2.2. *Let $\sigma, \sigma' \in \mathcal{H}(k)$. Then the closures of σ and σ' are link-homotopic as ordered, oriented links if and only if there is a sequence $\sigma = \sigma_0, \sigma_1, \ldots, \sigma_n = \sigma'$ of elements of $\mathcal{H}(k)$ such that σ_{j+1} is either a conjugation or a partial conjugation of σ_j.*

For a group G, we will denote by G_n the n^{th} term of the lower central series of G, i.e. $G_1 = G$ and $G_{n+1} = [G_n, G]$, the normal subgroup of G generated by elements of the form $[g,h] = ghg^{-1}h^{-1}$ for all $g \in G_n$ and $h \in G$. A group G is nilpotent of class n if $G_{n+1} = 1$ but $G_n \neq 1$. We summarize some known facts about the group structures of $\mathcal{H}(k)$ in the following lemma.

LEMMA 2.3. 1) *$\mathcal{H}(k)$ is torsion free and nilpotent of class $k-1$.*

2) *Corresponding to a decomposition $\mathcal{H}(k) = \mathcal{H}(k-1) \ltimes RF(k-1)$, we have*

$$H(k)_n = \mathcal{H}(k-1)_n \ltimes RF(k-1)_n .$$

3) *$\mathcal{H}(k)_{n-1}/\mathcal{H}(k)_n$ is a free abelian group of rank $(n-2)!\binom{k}{n}$.*

For $\sigma \in \mathcal{H}(k)$, its image in $\mathcal{H}(k)/\mathcal{H}(k)_3$ can be described by $\binom{k}{2} + \binom{k}{3}$ integers. These integers are linking numbers l_{ij}, for $1 \leq i < j \leq k$, and Milnor's triple linking numbers $\mu(rst)$, for $1 \leq r < s < t \leq k$. We want to have them defined precisely and understand how they change when σ is changed by a conjugation or a partial conjugation.

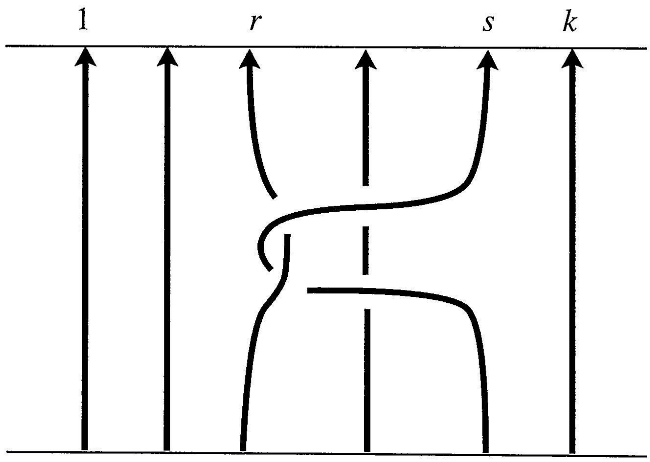

FIGURE 1

The pure braid τ_{rs}

We will denote by $\tau_{rs} = \tau_{sr}$, for $1 \leq r < s \leq k$, the pure braid depicted in Figure 1. Let $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$. For $1 \leq r < s < t \leq k$, after deleting all components other than the r, s, t-th components, σ can be written in the following normal form

$$\sigma = \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta}\,, \tag{2}$$

where $\alpha = l_{rs}$, $\beta = l_{rt}$, $\gamma = l_{st}$. By definition, we have $\delta = \mu(rst)$ for $\sigma \in \mathcal{H}(k)$.

LEMMA 2.4. *In $\mathcal{H}(k)/\mathcal{H}(k)_3$, if r', s', t' is a permutation of r, s, t and ϵ is the sign of the permutation, then*

$$[\tau_{r't'}, \tau_{s't'}] = [\tau_{rt}, \tau_{st}]^{\epsilon}\,.$$

Furthermore, we have

$$[\tau_{rt}^{\eta}, \tau_{st}] = [\tau_{rt}, \tau_{st}]^{\eta}\,.$$

This lemma is useful in the following calculation and its proof is straightforward.

To understand how $\mu(rst)$ changes under the conjugation, we only need to calculate the conjugation of $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ under the normal form (2) by $\tau_{rs}, \tau_{rt}, \tau_{st}$. This calculation is straightforward:

$$\begin{aligned}
\tau_{rs}\sigma\tau_{rs}^{-1} &= \tau_{rs}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta}\tau_{rs}^{-1} \\
&= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rs}, \tau_{rt}]^{\beta}[\tau_{rs}, \tau_{st}]^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta} \\
&= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta+\beta-\gamma}\,; \\
\tau_{rt}\sigma\tau_{rt}^{-1} &= \tau_{rt}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta}\tau_{rt}^{-1} \\
&= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{rs}]^{\alpha}[\tau_{rt}, \tau_{st}]^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta} \\
&= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta-\alpha+\gamma}\,; \\
\tau_{st}\sigma\tau_{st}^{-1} &= \tau_{st}\tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta}\tau_{st}^{-1} \\
&= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{st}, \tau_{rs}]^{\alpha}[\tau_{st}, \tau_{rt}]^{\beta}[\tau_{rt}, \tau_{st}]^{\delta} \\
&= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt}, \tau_{st}]^{\delta+\alpha-\beta}\,.
\end{aligned}$$

We summarize the calculation into the following lemma.

LEMMA 2.5. *The change of $\mu(rst)$ under a conjugation is given by*

Conjugation by τ_{rs}: $\quad \mu(rst) \to \mu(rst) + l_{rt} - l_{st}\,;$

Conjugation by τ_{rt}: $\quad \mu(rst) \to \mu(rst) - l_{rs} + l_{st}\,;$

Conjugation by τ_{st}: $\quad \mu(rst) \to \mu(rst) + l_{rs} - l_{rt}\,.$

Furthermore, $\mu(rst)$ will not change under a conjugation by τ_{ij} where $\{i,j\}$ and $\{r,s,t\}$ have at most one element in common.

The calculation of partial conjugations is slightly more complicated. We will start with partial conjugations by τ_{rt} and τ_{st}. These two operations are denoted by $\mathbf{t}^r$ and $\mathbf{t}^s$, respectively. For $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ under the normal form (2), we have:

$$\sigma \xrightarrow{\mathbf{t}^r} \tau_{rs}^{\alpha}\tau_{rt}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{rt}^{-1}$$
$$= \tau_{rs}\alpha\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\gamma};$$
$$\sigma \xrightarrow{\mathbf{t}^s} \tau_{rs}^{\alpha}\tau_{st}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta}\tau_{st}^{-1}$$
$$= \tau_{rs}\alpha\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta-\beta}.$$

To calculate partial conjugations by τ_{rs} and τ_{ts}, which are denoted by $\mathbf{s}^r$ and $\mathbf{s}^t$, respectively, we need to rewrite σ as follows:

$$\sigma = \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta} = \tau_{rt}^{\beta}\tau_{rs}^{\alpha}\tau_{ts}^{\gamma}[\tau_{rs},\tau_{ts}]^{-\delta-\alpha\beta}.$$

Then, we have:

$$\sigma \xrightarrow{\mathbf{s}^r} \tau_{rt}^{\beta}\tau_{rs}\tau_{rs}^{\alpha}\tau_{ts}^{\gamma}[\tau_{rs},\tau_{ts}]^{-\delta-\alpha\beta}\tau_{rs}^{-1}$$
$$= \tau_{rt}^{\beta}\tau_{rs}^{\alpha}\tau_{ts}^{\gamma}[\tau_{rs},\tau_{ts}]^{-\delta-\alpha\beta+\gamma}$$
$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta-\gamma};$$
$$\sigma \xrightarrow{\mathbf{s}^t} \tau_{rt}^{\beta}\tau_{ts}\tau_{rs}^{\alpha}\tau_{ts}^{\gamma}[\tau_{rs},\tau_{ts}]^{-\delta-\alpha\beta}\tau_{ts}^{-1}$$
$$= \tau_{rt}^{\beta}\tau_{rs}^{\alpha}\tau_{ts}^{\gamma}[\tau_{rs},\tau_{ts}]^{-\delta-\alpha\beta-\alpha}$$
$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\gamma}.$$

Similarly, to calculate partial conjugations $\mathbf{r}^s$ and $\mathbf{r}^t$, we first rewrite σ:

$$\sigma = \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta} = \tau_{st}^{\gamma}\tau_{sr}^{\alpha}\tau_{tr}^{\beta}[\tau_{sr},\tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma}.$$

Then, we have

$$\sigma \xrightarrow{\mathbf{r}^s} \tau_{st}^{\gamma}\tau_{sr}\tau_{sr}^{\alpha}\tau_{tr}^{\beta}[\tau_{sr},\tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma}\tau_{sr}^{-1}$$
$$= \tau_{st}^{\gamma}\tau_{sr}^{\alpha}\tau_{tr}^{\beta}[\tau_{sr},\tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma+\beta}$$
$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta+\beta};$$
$$\sigma \xrightarrow{\mathbf{r}^t} \tau_{st}^{\gamma}\tau_{tr}\tau_{sr}^{\alpha}\tau_{tr}^{\beta}[\tau_{sr},\tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma}\tau_{tr}^{-1}$$
$$= \tau_{st}^{\gamma}\tau_{sr}^{\alpha}\tau_{tr}^{\beta}[\tau_{sr},\tau_{tr}]^{\delta-\alpha\gamma+\beta\gamma-\alpha}$$
$$= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{\delta-\alpha}.$$

We summarize the previous calculation into the following lemma.

LEMMA 2.6. *The change of* $\mu(rst)$ *under a partial conjugation is given by*

$$\begin{aligned}
\mathbf{t}^r:&\quad \mu(rst) \to \mu(rst) + l_{st}\,;\\
\mathbf{t}^s:&\quad \mu(rst) \to \mu(rst) - l_{rt}\,;\\
\mathbf{s}^r:&\quad \mu(rst) \to \mu(rst) - l_{st}\,;\\
\mathbf{s}^t:&\quad \mu(rst) \to \mu(rst) + l_{rs}\,;\\
\mathbf{r}^s:&\quad \mu(rst) \to \mu(rst) + l_{rt}\,;\\
\mathbf{r}^t:&\quad \mu(rst) \to \mu(rst) - l_{rs}\,.
\end{aligned}$$

Furthermore, a partial conjugation by $\mathbf{i}^j$ *will not change* $\mu(rst)$ *if* $\{i,j\}$ *and* $\{r,s,t\}$ *have at most one element in common.*

For a given string link $\sigma \in \mathcal{H}(k)$, we will think of the whole collection $\{\mu(rst)\,;\ 1 \le r < s < t \le k\}$ as an element in $\mathbf{Z}^{\binom{k}{3}}$. Then the conjugations and partial conjugations act on $\mathbf{Z}^{\binom{k}{3}}$ by translations. We will abuse the notation by using the same symbol to denote both a translation operation and the corresponding translation vector. Thus, a translation operation $T\colon V \to V$ on a vector space V is given by $T(v) = v + T$, for all $v \in V$ and a fixed $T \in V$. If T_1 and T_2 are two translations, we have

$$(T_1 \cdot T_2)(v) = v + T_1 + T_2, \qquad \text{for all } v \in V\,.$$

The following two theorems follow directly from Lemmas 2.5 and 2.6.

THEOREM 2.7. *The translation operation on* $\mathbf{Z}^{\binom{k}{3}}$ *given by the conjugation of* τ_{ij} *is the same as the composition of the translation operations given by the partial conjugations* $\mathbf{i}^j$ *and* $\mathbf{j}^i$, *i.e. it is equal to* $\mathbf{i}^j + \mathbf{j}^i$.

THEOREM 2.8. *The translation operations* $\mathbf{i}^j$ *satisfy the following relations:*

$$\sum_{j \ne i} \mathbf{j}^i = 0\,, \qquad \sum_{j \ne i} l_{ij}\,\mathbf{i}^j = 0$$

for all $i = 1, 2, \ldots, k$.

String links are oriented in the sense that each component is given an orientation from the bottom to the top. See Figure 1. Reversing the orientation on each component of a string link defines a bijection

$$\sigma \mapsto \overline{\sigma}\colon \mathcal{H}(k) \to \mathcal{H}(k)\,.$$

This bijection is an anti-homomorphism: $\overline{\sigma_1\sigma_2} = \overline{\sigma}_2\,\overline{\sigma}_1$. This bijection induces an operation on $\mathbf{Z}^{\binom{k}{3}}$.

THEOREM 2.9. *The operation on $\mathbf{Z}^{\binom{k}{3}}$ induced by reversing the orientation of each component of a string link is to change each $\mu(rst)$ to $-\mu(rst)$ followed by the translation operation*

$$\mu(rst) \longrightarrow \mu(rst) - l_{rs}\,l_{rt} + l_{rs}\,l_{st} - l_{rt}\,l_{st}\,.$$

Proof. Consider the normal form (2) of $\sigma \in \mathcal{H}(k)/\mathcal{H}(k)_3$ in the r,s,t-th components. The normal form for $\overline{\sigma}$ is obtained as follows:

$$\begin{aligned}\overline{\sigma} &= [\tau_{rt},\tau_{st}]^{-\delta}\tau_{st}^{\gamma}\tau_{rt}^{\beta}\tau_{rs}^{\alpha}\\ &= \tau_{rs}^{\alpha}\tau_{rt}^{\beta}\tau_{st}^{\gamma}[\tau_{rt},\tau_{st}]^{-\delta-\alpha\beta+\alpha\gamma-\beta\gamma}\,.\end{aligned}$$

Thus the operation on $Z^{\binom{k}{3}}$ induced by $\sigma \mapsto \bar{\sigma}$ is given by

$$\mu(rst) \longrightarrow -\mu(rst) - l_{rs}\,l_{rt} + l_{rs}\,l_{st} - l_{rt}\,l_{st}\,. \qquad \square$$

3. CONSTRUCTION OF THE INVARIANT

By Theorems 2.2 and 2.7, we shall look for polynomials in l_{ij} and $\mu(rst)$ invariant under the translation operations on $\{\mu(rst)\} \in \mathbf{Z}^{\binom{k}{3}}$ induced by partial conjugations. There are $k(k-1)$ partial conjugations altogether and their induced translations subject to $2k$ linear equations given in Theorem 2.8. If these equations are linearly independent for generic values of $\{l_{ij}\}$, the sublattice of $Z^{\binom{k}{3}}$ generated by the translation vectors of the partial conjugations will be of dimension no larger than $k(k-1) - 2k = k^2 - 3k$.

LEMMA 3.1. *For $k > 3$, the $2k$ equations in Theorem 2.8 are linearly independent for generic values of $\{l_{ij}\}$.*

Proof. We write the two sets of equations in Theorem 2.8 as follows:

$$\mathbf{1}^i + \mathbf{2}^i + \cdots + \mathbf{j}^i + \cdots + \mathbf{k}^i = 0, \quad j \neq i;$$

$$l_{i1}\mathbf{i}^1 + l_{i2}\mathbf{i}^2 + \cdots + l_{ij}\mathbf{i}^j + \cdots + l_{ik}\mathbf{i}^k = 0, \quad j \neq i,$$

for each $i = 1, 2, \ldots, k$.

For generic values of $\{l_{ij}\}$, using the first $k-1$ equations from the first set of k equations, we can solve for $\mathbf{k}^1, \mathbf{k}^2, \ldots, \mathbf{k}^{k-1}$. Similarly, we can solve