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## 2. SOME CLASSICAL DEFINITIONS

We begin with some very general definitions concerning group actions. For an introduction to this subject, we refer to [42].

Let  $\Gamma$  be any group and  $X$  be any topological space. An *action* of  $\Gamma$  on  $X$  is a homomorphism  $\phi$  from  $\Gamma$  to the group  $\text{Homeo}(X)$  of homeomorphisms of  $X$ . An element  $\gamma \in \Gamma$  and a point  $x \in X$  produce the point  $\gamma \cdot x = \phi(\gamma)(x)$ . Conversely a map

$$(\gamma, x) \in \Gamma \times X \mapsto \gamma \cdot x \in X$$

comes from an action if for every  $\gamma$ , the point  $\gamma \cdot x$  depends continuously on  $x$  and if for every  $\gamma_1, \gamma_2$  we have  $\gamma_1 \cdot (\gamma_2 \cdot x) = (\gamma_1 \gamma_2) \cdot x$  and  $e \cdot x = x$  ( $e$  denotes the identity element in  $\Gamma$ ).

Two actions  $\phi_1$  and  $\phi_2$  of  $\Gamma$  on  $X_1$  and  $X_2$  are *conjugate* if there exists a homeomorphism  $h$  from  $X_1$  to  $X_2$  such that for every  $\gamma \in \Gamma$ , one has  $\phi_2(\gamma) = h\phi_1(\gamma)h^{-1}$ .

An action  $\phi$  is *faithful* if it is injective, *i.e.* if non trivial elements in the group act non trivially on the space. This is a minor assumption since we can always consider the associated faithful action of the quotient group  $\Gamma/\ker(\phi)$ .

The *orbit* of a point  $x$  is the set  $\mathcal{O}(x) = \{\phi(\gamma)(x) \mid \gamma \in \Gamma\} \subset X$ . The main object of topological dynamics is to study the topological properties of the partition of  $X$  into orbits. An action is *transitive* if there is only one orbit. We say in this case that  $X$  is *homogeneous* under the action of  $\Gamma$ . Of course, these transitive actions are quite trivial from the topological dynamics point of view but this does not mean that the geometrical study of homogeneous spaces is not interesting!

The *stabilizer* of the point  $x$  is the subgroup

$$\text{Stab}(x) = \{\gamma \in \Gamma \mid \phi(\gamma)(x) = x\} \subset \Gamma.$$

There is a natural bijection between the quotient  $\Gamma/\text{Stab}(x)$  and the orbit  $\mathcal{O}(x)$ . Note that the stabilizers of two points in the same orbit are conjugate subgroups in  $\Gamma$ . An action is *free* if the stabilizer of every point is trivial, *i.e.* if the action of a non trivial element of  $\Gamma$  has no fixed point.

In some cases,  $\Gamma$  might be a topological group. In these cases, we frequently consider *continuous actions* such that  $\gamma \cdot x$  is a continuous function on  $\Gamma \times X$ . The orbit map bijection from  $\Gamma/\text{Stab}(x)$  to  $\mathcal{O}(x)$  is continuous but is usually not a homeomorphism when  $\mathcal{O}(x)$  is equipped with the induced topology from  $X$ . The easiest non trivial example is the case where  $\Gamma = \mathbf{R}$ , *i.e.* of a topological flow: if the stabilizer of a point  $x$  is trivial, the orbit  $\mathcal{O}(x)$  is the image of a continuous bijection  $\mathbf{R} \rightarrow \mathcal{O}(x) \subset X$  but in many cases this orbit might be recurrent (for instance dense in  $X$ ) and this bijection is not a homeomorphism. There is however a special case in which this map is indeed a homeomorphism and we use this fact constantly (and sometimes implicitly) in these notes. Consider a Lie group  $G$  acting continuously and transitively on a manifold  $M$  and denote by  $H$  the stabilizer of a point. Then  $H$  is a closed subgroup of  $G$ , hence a closed Lie subgroup, and the quotient space  $G/H$  is naturally a smooth manifold. In this case, the orbit map from  $G/H$  to  $M$  is a homeomorphism.