4.1 Locally compact groups acting on the circle

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does not contain any non abelian free subgroups. Indeed, it is very easy to find free subgroups on two generators in $\mathrm{PL}_+(\mathbf{S}^1)$ using for instance the classical "Klein ping-pong lemma" (see [31] or Section 5.2). Later in this paper, we shall prove that "most subgroups" of $\mathrm{Homeo}_+(\mathbf{S}^1)$ contain free subgroups (5.14).

4.1 Locally compact groups acting on the circle

Recall that a very important (and difficult) theorem of Montgomery and Zippin states that a locally compact group is a Lie group if and only if there is a neighbourhood of the identity which does not contain a non trivial compact subgroup [40, 56]. We know the structure of compact subgroups of $\mathrm{Homeo}_+(\mathbf{S}^1)$: they are conjugate to subgroups of $\mathrm{SO}(2,\mathbf{R})$ and therefore they are either finite cyclic groups or conjugate to $\mathrm{SO}(2,\mathbf{R})$. None of these subgroups can be in a small neighbourhood of the identity. Indeed, consider the neighbourhood U of the identity in $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ consisting of those homeomorphisms f such that the distance between x and $f(x)$ is less than $1/3$ for all x in $\mathbf{R}/\mathbf{Z}$. Every element f in U has a unique lift $\widetilde{f}$ in $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$ which is such that $|\widetilde{f}(\widetilde{x})-\widetilde{x}| \leq 1/3$ for all $\widetilde{x}$ in $\mathbf{R}$. Of course, if f, g and fg are in U, we have $\widetilde{fg} = \widetilde{f}\widetilde{g}$. In particular, if there were a non trivial subgroup H contained in U this subgroup H would lift as a subgroup of $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$. Since $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$ is a torsion free group and since any compact subgroup of $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ contains elements of finite order, it follows that no non trivial compact subgroup of $\mathrm{Homeo}_+(\mathbf{R}/\mathbf{Z})$ can lift to $\widetilde{\mathrm{Homeo}}_+(\mathbf{R}/\mathbf{Z})$. In particular U contains no non trivial compact subgroup. We deduce:

THEOREM 4.7. *A locally compact subgroup of* $\mathrm{Homeo}_+(\mathbf{S}^1)$ *is a Lie group.*

It would be interesting to prove this theorem by elementary means, *i.e.* without the use of the Montgomery-Zippin theorem.

Consider a *connected Lie group G acting continuously and faithfully* on the circle by a homomorphism $\phi\colon G \to \mathrm{Homeo}_+(\mathbf{S}^1)$. Our objective is to determine all such actions. Orbits of the action are connected, so they can be of three kinds: the full circle, an open interval or a point. In other words, there is a closed set $F \subset \mathbf{S}^1$ (which might be empty) consisting of fixed points for the action, and the orbits which are not fixed points are the connected components of $\mathbf{S}^1 - F$. So, in order to understand the action, it is basically sufficient to understand it on each 1-dimensional orbit (homeomorphic to $\mathbf{R}$ or $\mathbf{S}^1$). Note that the action of G on one orbit is not necessarily faithful anymore but, taking the quotient by the kernel, we are led to study transitive and faithful actions of a connected Lie group G on $\mathbf{R}$ or $\mathbf{S}^1$.

Denote by H the stabilizer of a point in such an orbit. This is a closed subgroup of G, hence a Lie subgroup of codimension 1 and G acts smoothly on the 1-dimensional manifold G/H. The Lie algebra $\mathfrak{G}$ will therefore induce a finite dimensional Lie algebra of smooth vector fields on G/H. Since G acts transitively on G/H, for any point on G/H there is an element of this Lie algebra which does not vanish at this point.

Consider the case of the projective action of $\mathrm{PSL}(2,\mathbf{R})$ on $\mathbf{RP}^1$. The Lie algebra $\mathfrak{sl}(2,\mathbf{R})$ is the algebra of 2×2 real matrices with trace 0. Taking the differential of the action at the identity, one easily checks that the corresponding Lie algebra of vector fields is the algebra of vector fields of the form $u(x)\partial/\partial x$ where u is a polynomial of degree at most 2; thus we get the following identification of algebras:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2,\mathbf{R}) \mapsto (b+2ax-cx^2)\frac{\partial}{\partial x}\,.$$

Denote by $\mathfrak{Vect}$ the Lie algebra of germs of smooth vector fields of $\mathbf{R}$ in the neighbourhood of 0. The subspace $\mathfrak{Vect}_k$ of vector fields $u(x)\partial/\partial x$ where u vanishes at the origin together with its first k derivatives is an ideal in $\mathfrak{Vect}$ and the quotient Lie algebra $\mathfrak{Vect}/\mathfrak{Vect}_k$ can be identified, as a vector space with the space $\mathfrak{P}_k$ of vector fields of the form $u(x)\partial/\partial x$ where u is a polynomial of degree at most k.

Note however that $\mathfrak{P}_k$ is a subalgebra of $\mathfrak{Vect}$ if and only if $k=0,1$ or 2. One can therefore think of $\mathfrak{P}_0,\mathfrak{P}_1,\mathfrak{P}_2$ at the same time as subalgebras of $\mathfrak{Vect}$ and as quotient algebras of $\mathfrak{Vect}$.

The general situation was analyzed a long time ago by Lie, who found all the possibilities [45]:

THEOREM 4.8 (Lie). *Let $\mathfrak{G}$ be a non trivial finite dimensional Lie algebra consisting of germs of smooth vectors fields in the neighbourhood of* 0 *in* $\mathbf{R}$. *Assume that not all these vector fields vanish at the origin. Then the dimension of $\mathfrak{G}$ is at most 3. More precisely, in suitable coordinates $\mathfrak{G}$ consists of all germs of the form $u(x)\partial/\partial x$ where u is a polynomial of degree less than or equal to k for $k=0,1$ or* 2.

Proof. Since one element of $\mathfrak{G}$ does not vanish at the origin, we can find a suitable local coordinate x such that the germ of this element is $\partial/\partial x$. Let $\mathcal{E}$ be the finite dimensional vector space of germs of functions u such that $u(x)\partial/\partial x$ belongs to $\mathfrak{G}$. Of course $\mathcal{E}$ contains the constants and is stable under the operation of taking derivatives, since the bracket $[\partial/\partial x, u(x)\partial/\partial x]$ equals $u'(x)\partial/\partial x$. The successive iterates of the linear operator induced by the derivative must be linearly dependent. This shows that there exists a linear differential equation with constant coefficients which is satisfied by all elements in $\mathcal{E}$. It follows that all elements in $\mathcal{E}$ are real analytic functions. Every non trivial element u of $\mathcal{E}$ therefore has a convergent Taylor expansion of the form $u(x) = a_i x^i + \dots$ with $a_i \neq 0$. Moreover, this integer i is bounded since a solution of a linear differential equation with constant coefficients which vanishes at a point together with its derivatives of orders up to the degree of the equation has to vanish identically. Choose an element u for which the integer i is maximal. Now the algebra $\mathfrak{G}$ contains $[u(x)\partial/\partial x, u'(x)\partial/\partial x] = a_i(x^{2i-2} + \dots)\partial/\partial x$. It follows that $2i-2 \leq i$, so that $i \leq 2$.

For each element of $\mathfrak{G}$, consider the Taylor expansion of degree 2 of the associated vector field, considered as an element of $\mathfrak{P}_2 \simeq \mathfrak{sl}(2,\mathbf{R})$. This produces a linear map $j_2 \colon \mathfrak{G} \to \mathfrak{P}_2$ which is clearly an algebra homomorphism and which is injective by the previous argument.

If the image of j_2 is 1-dimensional, then $\mathfrak{G}$ consists only of constant multiples of $\partial/\partial x$. In this case, G is (locally) isomorphic to $\mathbf{R}$ and the Lie algebra of H is trivial, which means that H is discrete.

Suppose that the image of j_2 is 3-dimensional, *i.e.* that j_2 is an isomorphism. Consider the element $X = \partial/\partial x$ of $\mathfrak{P}_2$. Note that the linear operator $ad^3(X) \colon \mathfrak{P}_2 \to \mathfrak{P}_2$ is trivial. The vector field ${j_2}^{-1}(X)$ does not vanish at the origin so that we could have used it at the beginning when we chose a local coordinate x. In other words, there is a local coordinate x such that $\partial/\partial x$ belongs to $\mathfrak{G}$ and such that the linear operator induced by taking bracket with $\partial/\partial x$ is nilpotent of order 3. This means that the third derivative of every element of $\mathcal{E}$ vanishes. In suitable coordinates $\mathfrak{G}$ coincides with polynomial vector fields of degree at most 2. In this case, G is locally isomorphic to $\mathrm{SL}(2,\mathbf{R})$ and H is locally isomorphic to the group of upper triangular matrices.

Suppose finally that the image of j_2 is 2-dimensional. This means that the Taylor expansion of order 1 is an isomorphism $j_1 \colon \mathfrak{G} \to \mathfrak{P}_1$ and one can reproduce the above proof with the nilpotent operator of order 2 induced by $\partial/\partial x$. In this case, G is locally isomorphic to the 2-dimensional group of upper triangular matrices in $\mathrm{SL}(2,\mathbf{R})$ and H is locally isomorphic to the 1-dimensional subgroup of unipotent matrices. □

This theorem gives a complete *local* description of transitive actions of a Lie group. It is not difficult to deduce the complete classification of transitive and faithful actions of connected Lie groups on 1-manifolds. Up to conjugacy, the list is the following.

- The action of $\mathbf{R}$ on itself.
- The action of the circle $\mathbf{R}/\lambda\mathbf{Z}$ on itself (for $\lambda > 0$).
- The action of the affine group $\mathrm{Aff}_+(\mathbf{R})$ on $\mathbf{R}$.
- The action of the k-fold cover $\mathrm{PSL}_k(2,\mathbf{R})$ of $\mathrm{PSL}(2,\mathbf{R})$ on the circle, described in Section 4 (for $k \geq 1$).
- The action of the universal cover $\widetilde{\mathrm{SL}}(2,\mathbf{R})$ of $\mathrm{SL}(2,\mathbf{R})$ on the universal cover of $\mathbf{S}^1$.

Loosely speaking, we could say that there are three geometries of finite type on 1-manifolds: euclidean, affine and projective.

The full description of faithful non transitive actions of a connected Lie group G on the circle is now easy in principle. We should choose a closed set $F \subset \mathbf{S}^1$ consisting of fixed points and for each connected component I of the complement of F, the action is described by some surjection from G to $\mathbf{R}$, $\mathrm{Aff}_+(\mathbf{R})$, $\mathrm{PSL}_k(2,\mathbf{R})$ or $\widetilde{\mathrm{SL}}(2,\mathbf{R})$.

As a trivial example, we get the description of *topological flows on the circle*, *i.e.* of actions of $\mathbf{R}$ on the circle. If such a flow is transitive, it is conjugate to the action of $\mathbf{R}$ on the circle $\mathbf{R}/\lambda\mathbf{Z}$ for some $\lambda > 0$ (the "period" of the flow). If it is not transitive, it has a non empty set of fixed points $F \subset \mathbf{S}^1$ and the conjugacy class is completely described by the orientation: on each component of $\mathbf{S}^1 - F$, the flow is positive or negative.

Finally, we should describe the actions of non connected Lie groups G. Let G_0 be the connected component of the identity in G so that we already understand the action of G_0. Observe that G_0 is a normal subgroup of G so that the action of G preserves F and permutes the connected components of $\mathbf{S}^1 - F$. It is not easy to fully analyze this situation but it is quite clear that when G_0 is non trivial, its normalizer is usually very small. We leave to the reader the details of this analysis. Of course, when G_0 is trivial, *i.e.* when G is discrete, the previous discussion has no content. Hence among locally compact groups acting on the circle, the most interesting ones are the discrete groups.