# 6.1 Group cohomology

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of $\Gamma$ since, once again, $\Gamma$ commutes with $\theta$.

We observe that this new group of homeomorphisms of a circle satisfies (MINIMALITY) and (STRONG EXPANSIVITY). Minimality is obviously inherited from the same property of $\Gamma$ on $\mathbf{S}^1$. As for (STRONG EXPANSIVITY), it suffices to observe that any compact interval contained in $[x, \theta(x)[$ is contractible, by definition. This means that any compact interval in $\mathbf{S}^{1'}$ is contractible and this implies (STRONG EXPANSIVITY).

*We have now proved that if* (MINIMALITY) *and* (EXPANSIVITY) *are both satisfied, then the group* $\Gamma$ *must contain a free non abelian subgroup.*

Now, let us look more closely at (EXPANSIVITY) and observe that the negation of this property is nothing more than the *equicontinuity* property of the group $\Gamma$. If a group $\Gamma$ acts equicontinuously, then its closure in $\text{Homeo}_+(\mathbf{S}^1)$ is a compact group by Ascoli's theorem. We analyzed compact subgroups of $\text{Homeo}_+(\mathbf{S}^1)$ in 4.1 : they turned out to be abelian and conjugate to groups of rotations.

*We have shown that if* (MINIMALITY) *holds then* $\Gamma$ *is either abelian or contains a free non abelian subgroup; in other words, we have proved Corollary 5.15.*

Proving Theorem 5.14 in full generality is now an easy matter. Let $\Gamma$ be any subgroup of $\text{Homeo}_+(\mathbf{S}^1)$ and let us use the structure theorem 5.6–5.8. If $\Gamma$ is minimal, we have already proved the theorem. If $\Gamma$ has a finite orbit, there is a $\Gamma$-invariant probability which is a finite sum of Dirac masses. Finally, if there is an exceptional minimal set, the $\Gamma$-action is semi-conjugate to a minimal action. Applying our proof to this minimal action, we deduce that $\Gamma$ contains a non abelian free subgroup unless the restriction of the action of $\Gamma$ to the exceptional minimal set is abelian and is semi-conjugate to a group of rotations. In this case, one finds a $\Gamma$-invariant measure whose support is the exceptional minimal set. This is the end of the proof of Theorem 5.14.

## 6. BOUNDED EULER CLASS

### 6.1 GROUP COHOMOLOGY

Let us begin this section with some algebra. Let $\Gamma$ be any group. Let us consider the (semi)-simplicial set $E\Gamma$ whose vertices are the elements of $\Gamma$ and for which $n$-simplices are all $(n+1)$-tuples of elements of $\Gamma$. The $i^{\text{th}}$ face of the simplex $(\gamma_0, \dots, \gamma_k)$ is $(\gamma_0, \dots, \hat{\gamma}_i \dots \gamma_k)$ where the term $\gamma_i$ is omitted. Note that the set $E\Gamma$ does not depend on the group structure of $\Gamma$.

As a matter of fact, $E\Gamma$ is contractible since it is the full simplex over the set $\Gamma$. However, there is a simplicial free action of $\Gamma$ on $E\Gamma$ induced by left translations of $\Gamma$ on itself. Hence once could think of the quotient $B\Gamma$ of $E\Gamma$ by this action as a space whose fundamental group is $\Gamma$ and with vanishing higher homotopy groups. One would like to define the cohomology of the group $\Gamma$ as the cohomology of this quotient space $B\Gamma$. We should be careful with $B\Gamma$ since it has only one vertex (a group acts transitively on itself !).

However, guided by this idea, it is natural to define a *k-cochain* of $\Gamma$ with values in some abelian group $A$ as a map $c\colon \Gamma^{k+1} \to A$ which is *homogeneous, i.e.* such that $c(\gamma\gamma_0, \gamma\gamma_1, \dots, \gamma\gamma_k) = c(\gamma_0, \gamma_1, \dots, \gamma_k)$ identically. The set of these cochains is an abelian group denoted by $C^k(\Gamma, A)$. We have a natural coboundary $d_k$ from $C^k(\Gamma, A)$ to $C^{k+1}(\Gamma, A)$ defined by

$$d_k c(\gamma_0, \dots, \gamma_{k+1}) = \sum_{i=0}^{k} (-1)^i c(\gamma_0, \dots, \widehat{\gamma_i}, \dots, \gamma_k) .$$

Of course, we have $d_{k+1} \circ d_k = 0$ and we define the *cohomology group* $H^k(\Gamma, A)$ as being the quotient of cocycles (*i.e.* the kernel of $d_k$) by coboundaries (*i.e.* the image of $d_{k-1}$). If $A$ is moreover a ring, then there is a natural cup product from $H^k(\Gamma, A) \times H^l(\Gamma, A)$ to $H^{k+l}(\Gamma, A)$. We refer to [11] for an excellent account of this theory of group cohomology. Note that for any homomorphism $\phi$ from a group $\Gamma$ to another group $\Gamma'$, there is an induced homomorphism $\phi^\star\colon H^k(\Gamma', A) \to H^k(\Gamma, A)$.

A homogeneous map $c\colon \Gamma^{k+1} \to A$ can be written in a unique way in the form $c(\gamma_0, \dots, \gamma_k) = \bar{c}(\gamma_0^{-1}\gamma_1, \gamma_1^{-1}\gamma_2, \dots, \gamma_{k-1}^{-1}\gamma_k)$ for a unique function $\bar{c}\colon \Gamma^k \to A$. Conversely, given a map $\bar{c}$ there is a unique homogeneous map $c$ satisfying this relation. One says that $\bar{c}$ is the inhomogeneous cochain associated to $c$. In other words, the space $C^k(\Gamma, A)$ is canonically isomorphic to the $A$-module of all maps $\Gamma^k \to A$.

In degree 1, a cochain is a homogeneous map $c\colon \Gamma^2 \to A$ and the corresponding inhomogeneous cochain is a map $\bar{c}\colon \Gamma \to A$. It is interesting to check that $c$ is a cocycle if and only if $\bar{c}$ is a homomorphism. Moreover 0-cochains are constant maps from $\Gamma$ to $A$ and their coboundary is therefore $0$. It follows that *for any group* $\Gamma$, *the cohomology* $H^1(\Gamma, A)$ *is identified with the set of homomorphisms from* $\Gamma$ *to* $A$.

In degree 2, the interpretation is quite interesting. Consider a central extension of $\Gamma$ by $A$ :

$$0 \longrightarrow A \xrightarrow{\ i\ } \widetilde{\Gamma} \xrightarrow{\ p\ } \Gamma \longrightarrow 1 .$$

This means that $\widetilde{\Gamma}$ contains a subgroup isomorphic to $A$ contained in its center and that the quotient by this subgroup is isomorphic to $\Gamma$. Suppose that the projection $p$ has a section $s$ which is a homomorphism from $\Gamma$ to $\widetilde{\Gamma}$ such that $p \circ s = Id_\Gamma$. Then it follows that $\widetilde{\Gamma}$ is isomorphic to the direct product $\Gamma \times A$ by the homomorphism sending $(\gamma, a)$ to $s(\gamma)i(a)$. Hence, in order to measure the non triviality of an extension we try to find the "obstruction" to finding a section $s$. This is done in the following way. Choose a set theoretical section $s$ from $\Gamma$ to $\widetilde{\Gamma}$; this is possible since $p$ is onto. If $\gamma_1$ and $\gamma_2$ are two elements of $\Gamma$, consider $\bar{c}(\gamma_1, \gamma_2) = s(\gamma_1\gamma_2)^{-1}s(\gamma_1)s(\gamma_2)$. This element projects on the identity element of $\Gamma$ under $p$ since $p$ is a homomorphism; it is therefore an element of the image of $i$ and can be identified with an element of $A$. This defines a map $\bar{c}\colon \Gamma^2 \to A$. Let $c\colon \Gamma^3 \to A$ be the associated homogeneous cochain. One checks that $c$ is a cocycle. Of course, the section $s$ is not unique but another choice $s'$ has the form $s'(\gamma) = s(\gamma)i(u(\gamma))$ for some function $u\colon \Gamma \to A$. If one computes the cocycle $c'$ associated to this new choice of a section $s'$, one finds that $c' - c$ is the coboundary of the 1-cochain associated to the map $u$. It follows that the cohomology class of $c$ in $H^2(\Gamma, A)$ is well defined, *i.e.* does not depend on the choice of a section. This cohomology class is called the *Euler class of the extension* under consideration.

It is not difficult to check the following properties of the Euler class.

1) Two central extensions $\widetilde{\Gamma}_1$ and $\widetilde{\Gamma}_2$ of $A$ by $\Gamma$ are isomorphic by some isomorphism which is the identity on the central subgroup $A$ and inducing the identity on the quotient $\Gamma$ if and only if they have the same Euler class in $H^2(\Gamma, A)$.

2) Any class in $H^2(\Gamma, A)$ corresponds to a central extension.

In short, $H^2(\Gamma, A)$ *parametrizes isomorphism classes of central extensions of* $A$ *by* $\Gamma$.

Before coming back to the dynamics of groups acting on the circle, let us consider a few simple examples.

If $\Gamma = \mathbf{Z}$, it is clear that every extension admits a section which is a homomorphism: it suffices to choose arbitrarily $s(1)$ in $p^{-1}(1)$ and to define $s(n) = s(1)^n$ for $n \in \mathbf{Z}$. Hence, if $\Gamma = \mathbf{Z}$ or more generally if $\Gamma$ is a free group, we have $H^2(\Gamma, A) = 0$.

Let $\Gamma_g$ be the fundamental group of a closed oriented surface of genus $g \geq 1$. It has a presentation of the form

$$\Gamma_g = \langle a_1, b_1, \dots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1 \rangle .$$

Now consider the group $\widetilde{\Gamma}_g$ defined by the presentation

$$\widetilde{\Gamma}_g = \big\langle z, a_1, b_1, \dots, a_g, b_g \mid \\ a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = z, \quad z a_i = a_i z, \quad z b_i = b_i z \big\rangle .$$

The central subgroup $A$ generated by $z$ turns out to be infinite cyclic so that $\widetilde{\Gamma}_g$ defines a central extension of $\Gamma_g$ by $\mathbf{Z}$, hence an Euler class in $H^2(\Gamma_g, \mathbf{Z})$. It is a fact that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic with $\mathbf{Z}$ and that the element that we have just constructed is a generator of this cohomology group. We shall not prove this here but we note that this is related to the fact that a closed oriented surface of genus $g \geq 1$ has a contractible universal cover and that the cohomology of $\Gamma_g$ can therefore be identified with the cohomology of the compact oriented surface of genus $g$ (see [11] for more details).

## 6.2 The Euler class of a group action on the circle

We have already met a central extension related to groups of homeomorphisms

$$0 \longrightarrow \mathbf{Z} \longrightarrow \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1) \xrightarrow{p} \mathrm{Homeo}_+(\mathbf{S}^1) \longrightarrow 1 .$$

The cohomology group $H^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{Z})$ has been computed. It is isomorphic to $\mathbf{Z}$ and a generator is the Euler class of this central extension [50].

Consider now a homomorphism $\phi$ from some group $\Gamma$ to $\mathrm{Homeo}_+(\mathbf{S}^1)$. Then, we can pull back the previous extension by $\phi$. In other words, we consider the set of $(\gamma, \widetilde{f}) \in \Gamma \times \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi(\gamma) = p(\widetilde{f})$. This is a group $\widetilde{\Gamma}$ equipped with a canonical projection onto $\Gamma$ whose kernel is isomorphic to $\mathbf{Z}$, *i.e.* $\widetilde{\Gamma}$ is a central extension of $\Gamma$ by $\mathbf{Z}$. In case $\phi$ is injective, $\widetilde{\Gamma}$ is just the pre-image of $\phi(\Gamma)$ under $p$, which is the group of lifts of $\phi(\Gamma)$. The Euler class of this central extension of $\Gamma$ is called *the Euler class of the homomorphism* $\phi$ and denoted by $eu(\phi) \in H^2(\Gamma, \mathbf{Z})$. It is obviously a dynamical invariant in the sense that two conjugate homomorphisms $\phi_1$ and $\phi_2$ have the same Euler class in $H^2(\Gamma, \mathbf{Z})$. Note that it follows from the definition that $eu(\phi)$ is zero if and only if the homomorphism $\phi$ lifts to a homomorphism $\widetilde{\phi}\colon \Gamma \to \widetilde{\mathrm{Homeo}}_+(\mathbf{S}^1)$ such that $\phi = p \circ \widetilde{\phi}$.

A few examples are in order. In the case of a single homeomorphism, *i.e.* when $\Gamma = \mathbf{Z}$, we saw that $H^2(\mathbf{Z}, \mathbf{Z}) = 0$. Hence the Euler class vanishes and our new invariant is very poor indeed: in particular, it does not detect the rotation number. A similar phenomenon occurs when $\Gamma$ is free.

If $\Gamma_g$ is the fundamental group of a closed oriented surface of genus $g \geq 1$, we know that $H^2(\Gamma_g, \mathbf{Z})$ is isomorphic to $\mathbf{Z}$ so that the Euler class