6.4 EXPLICIT BOUNDS ON THE EULER CLASS

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Now, we can state the general result which is the exact analogue of what has been done in 5.9 for the rotation number. We don't give the proof: it can be found in [25] (in a slightly different terminology and with small mistakes...), but the reader should now be in a condition to fill in the missing details by himself.

THEOREM 6.6 ([25]). *Let ϕ_1, ϕ_2 two homomorphisms from a group Γ to* $\mathrm{Homeo}_+(\mathbf{S}^1)$. *Assume that the bounded Euler classes $\phi_1^*(eu) = \phi_2^*(eu)$ are equal to the same class c in $H_b^2(\Gamma, \mathbf{Z})$.*

1) *If c is a rational class, then $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits with the same cyclic structure.*

2) *If c is not rational, then the associated minimal homomorphisms $\overline{\phi}_1$ and $\overline{\phi}_2$ are conjugate.*

Conversely, if $\phi_1(\Gamma)$ and $\phi_2(\Gamma)$ have finite orbits of the same cyclic structure or if they have no finite orbit and their associated minimal homomorphisms are conjugate (by an orientation preserving homeomorphism), then they have the same bounded Euler class.

Note in particular that the bounded Euler class of an action vanishes if and only if there is a point on the circle which is fixed by all the elements of the group.

6.4 EXPLICIT BOUNDS ON THE EULER CLASS

Since we know that the bounded Euler class of an action contains almost all the topological information, it is very natural to try to determine the part of $H_b^2(\Gamma, \mathbf{Z})$ which corresponds to the bounded Euler classes of all actions of Γ on the circle. In the case $\Gamma = \mathbf{Z}$, we know that $H_b^2(\mathbf{Z}, \mathbf{Z}) = \mathbf{R}/\mathbf{Z}$ and that every class corresponds to an action (by rotations). However, in the case where Γ is the fundamental group of a closed oriented surface of genus $g \geq 1$, the Milnor-Wood inequality shows that even the usual Euler class in $H^2(\Gamma, \mathbf{Z}) = \mathbf{Z}$ has to satisfy some inequality.

Given a bounded cochain c in $C_b^k(\Gamma, \mathbf{R})$, we define its norm $\|c\|$ as the supremum of the absolute value of $c(\gamma_0, \dots, \gamma_k)$. Then we define the "norm" of a bounded cohomology class with real coefficients as the infimum of the norms of cocycles that represent it. We should be aware of the fact that this norm is not really a norm but is merely a semi-norm: a non zero class might have zero norm... Consider the case of the bounded Euler class, seen in the real bounded cohomology.

THEOREM 6.7. *The image of the bounded Euler class eu in the real bounded cohomology* $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ *has norm* $1/2$.

Proof. This is the abstract version of the Milnor-Wood inequality. Note that a constant 2-cocycle is the coboundary of a constant 1-cochain. We found a representative of the Euler class taking only two values 0 and 1. If we subtract from this cocycle the constant cocycle taking the value $1/2$, we get a cohomologous bounded (real) cocycle taking values $\pm 1/2$. This shows that the norm of the image of *eu* in $H_b^2(\mathrm{Homeo}_+(\mathbf{S}^1), \mathbf{R})$ is at most $1/2$. The opposite inequality follows from Milnor's computation of the Euler number for an embedding of the fundamental group Γ_g of a closed oriented surface as a discrete cocompact subgroup of $\mathrm{PSL}(2, \mathbf{R})$ that we mentioned in 6.1. If the norm were strictly less than $1/2$, then this number would be strictly less than $2g - 2$. See [25] for more explanations. □

6.5 ACTIONS ON THE REAL LINE AND ORDERINGS

Our main concern is to study actions on the circle but there is a preliminary question which deals with actions on the line. Of course, if a group acts on the line, we can always add a point at infinity to produce an action on the circle (with a common fixed point). In other words studying actions on the line is equivalent to studying actions on the circle with vanishing bounded Euler class. This is the reason why we begin by general remarks on groups acting on the line.

Observe first that the dynamics of a single orientation preserving homeomorphism h of $\mathbf{R}$ are very easy to describe. Let $F = Fix(h)$ be the set of fixed points. Each interval of the complement of F is h-invariant and the action of h on this interval is conjugate to a translation (positive or negative, according to the sign of $h(x) - x$ on this interval).

We say that a group Γ is *left orderable* if there exists a *total* ordering $\preceq$ on Γ which is invariant under left translations (*i.e.* $\gamma_1 \preceq \gamma_2$ implies $\gamma\gamma_1 \preceq \gamma\gamma_2$). We write $\gamma_1 \prec \gamma_2$ if $\gamma_1 \preceq \gamma_2$ and $\gamma_1 \neq \gamma_2$. An obvious necessary condition for a group to be left orderable is that it be torsion free (*i.e.* there is no non trivial element of finite order).

The following theorem is well known but we weren't able to find its origin in the literature.