7.4 Some groups that do act...

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.08.2024**

corresponding map $\Psi_j^{(3)} : X_j \to \mathbf{S}^1_k$ and we show, as above, that the image of this map consists almost everywhere of triples of the form (A,A,A). It follows that for each $j = 1, \ldots, n$ and on a subset of full measure, the image of a flag by Ψ depends only on its projection by pr_j. Since this is true for every j, this means that Ψ is constant almost everywhere. This is a contradiction and finishes the proof of Theorem 7.4 for lattices in $\mathrm{SL}(n, \mathbf{R})$.

Of course, these proofs immediately generalize to lattices in complex or quaternionic special linear groups $\mathrm{SL}(n, \mathbf{C})$ and $\mathrm{SL}(3, \mathbf{H})$ (for $n \geq 3$).

7.4 Some groups that do act...

We saw that many higher rank lattices don't act on the circle. To conclude these notes, we give some more examples of "big" groups acting on the circle. Let Σ be a compact oriented surface of genus $g \geq 2$ and $x \in \Sigma$ be some base point. The fundamental group $\pi_1(\Sigma, x)$ is a classical example of a hyperbolic group in the sense of Gromov (see for instance [27]). The boundary of this group is a topological circle: indeed $\pi_1(\Sigma, x)$ acts freely and cocompactly on the Poincaré disc so that $\pi_1(\Sigma, x)$ is quasi-isometric to the Poincaré disc. Consequently, the automorphism group $\mathrm{Aut}(\pi_1(\Sigma, x))$ acts naturally on the circle. This action is very interesting and has been very much studied. See for instance [21]. Note that $\mathrm{Aut}(\pi_1(\Sigma, x))$ contains the group of inner conjugacies and that the quotient $\mathrm{Out}(\pi_1(\Sigma, x))$ is the *mapping class group* of the surface (*i.e.* the group of isotopy classes of homeomorphisms of the surface):

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \mathrm{Aut}(\pi_1(\Sigma, x)) \longrightarrow \mathrm{Out}(\pi_1(\Sigma, x)) \longrightarrow 1\,.$$

Fix an element f of infinite order in this mapping class group and consider the group Γ_f which is the inverse image of the group generated by f in the previous exact sequence. We have an exact sequence:

$$1 \longrightarrow \pi_1(\Sigma, x) \longrightarrow \Gamma_f \longrightarrow \mathbf{Z} \longrightarrow 1\,.$$

This group Γ_f is the fundamental group of the 3-manifold which fibers over the circle and whose monodromy is given by the class f. Thurston showed that if f is of pseudo-Anosov type, then this 3-manifold is hyperbolic. In particular, for such a choice of f, the group Γ_f embeds as a discrete cocompact subgroup of the isometry group of the hyperbolic 3-ball, isomorphic to $\mathrm{PSL}(2, \mathbf{C})$. This construction provides many examples of faithful actions of (rank 1) lattices on the circle. In [68] Thurston constructs faithful actions of the fundamental group of many hyperbolic 3-manifolds on the circle.

Suppose now that Σ has one boundary component $\partial\Sigma$. Choose the base point on the boundary and equip Σ with a metric with curvature -1 and totally geodesic boundary. The universal cover $\widetilde{\Sigma}$ of Σ is therefore identified with the complement in the Poincaré disc of a disjoint union of half spaces. On the boundary of the disc, these half spaces define an open dense subset Ω whose complement is a Cantor set K which is the boundary of the hyperbolic group $\pi_1(\Sigma, x)$. The union $\partial\widetilde{\Sigma} \cup K$ is a topological circle and if we collapse each connected component of $\partial\widetilde{\Sigma}$ to a point, this circle collapses to another circle that we denote by C. Choose also a base point $\widetilde{x}$ above x in the universal cover. Consider now the mapping class group Γ of Σ *i.e.* the group of homeomorphisms of Σ modulo isotopy. A homeomorphism f of Σ has a lift $\widetilde{f}$ to $\widetilde{\Sigma}$ which fixes the boundary component containing $\widetilde{x}$. This homeomorphism $\widetilde{f}$ extends continuously to $\partial\widetilde{\Sigma} \cup K$ and defines a homeomorphism $\bar{f}$ of the circle C. Note that if two homeomorphisms are isotopic, the two corresponding extensions agree on the Cantor set K. The connected component of $\partial\widetilde{\Sigma}$ containing $\widetilde{x}$ yields a base point $\bar{x}$ in C which is fixed by all homeomorphisms $\bar{f}$. Hence we can define an action of Γ on a line by letting f act via $\bar{f}$ on the line $C - \{\bar{x}\}$. Hence we proved (following an idea of Thurston) that the mapping class group of (Σ, x) acts (faithfully) on a line and is therefore left orderable.

We could also use the same idea for surfaces with several boundary components, for instance the sphere minus a finite number of discs. The corresponding mapping class groups turn into the so called braid groups. In this way we get interesting faithful actions of braid groups on the line, or equivalently total left orderings. It is interesting to note that these orderings were initially discovered from a completely different point of view by Dehornoy [16].

To conclude this paper, we would like to mention a rich family of group actions on the circle, coming from the theory of Anosov flows on 3-manifolds. Let M be a compact connected 3-manifold with no boundary and X a non singular smooth vector field on M. Denote by ϕ^t the flow generated by X. One says that ϕ^t is an *Anosov flow* if there is a continuous splitting of the tangent bundle TM as a sum of three line bundles $TM = \mathbf{R}X \oplus E^{ss} \oplus E^{uu}$ which are invariant under (the differential of) the flow ϕ^t and such that vectors in E^{uu} are expanded, and vectors in E^{ss} are contracted. More precisely, this means that for any riemannian metric on M, there are constants $C > 0$ and $\lambda > 0$ such that for any time $t > 0$ and vectors $v_{ss} \in E^{ss}$ and $v_{uu} \in E^{uu}$,

$$||d\phi^t(v_{ss})|| \leq C\exp(\lambda t)||v_{ss}||\,,$$

$$||d\phi^t(v_{uu})|| \geq C\exp(\lambda t)||v_{uu}||\,.$$

This kind of flow is rather abundant on 3-manifolds. The main example, which gave birth to the theory, is the geodesic flow of a compact surface with negative curvature, acting on the unit tangent bundle of the surface. We refer to [3, 22] for a general presentation of the theory including a bibliography. Starting from some Anosov flow and selecting a periodic orbit, one can perform a Dehn surgery on this closed curve. It turns out that if the surgery is positive, one can define a flow on the new manifold which is still of Anosov type. Using this construction, one constructs many examples. For instance, one can construct Anosov flows on some hyperbolic 3-manifolds (*i.e.* admitting a metric of constant negative curvature).

One of the main properties of Anosov flows is that they give rise to two codimension one foliations. Indeed, it has been shown by Anosov that there are two codimension one foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ whose leaves are everywhere tangent to $E^{ss} \oplus \mathbf{R}X$ and $E^{ss} \oplus \mathbf{R}X$. Verjovsky showed that if one lifts the flow ϕ^t to the universal cover $\widetilde{M}$ of M, the orbits of the resulting flow $\widetilde{\phi}^t$ are the fibers of a (trivial) fibration of $\widetilde{M}$ over a surface S (diffeomorphic to $\mathbf{R}^2$). Lifting the two foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ to $\widetilde{M}$, we get two foliations which project to two transverse foliations by curves $\widetilde{f}^u$ and $\widetilde{f}^s$ on the surface S. One says that the flow is $\mathbf{R}$-*covered* if the leaves of $\widetilde{f}^u$ are the fibers of a (trivial) fibration $p_u\colon S \to \mathbf{R}_u$ (where $\mathbf{R}_u$ is homeomorphic to $\mathbf{R}$). It follows that the leaves of $\widetilde{f}^s$ are also the fibers of a (trivial) fibration $p_s\colon S \to \mathbf{R}_s$. For instance, the geodesic flow on a negatively curved surface is $\mathbf{R}$-covered. It turns out that a positive surgery on an $\mathbf{R}$-covered Anosov flow is still $\mathbf{R}$-covered so that we get many examples. Consider the map $(p_u, p_s)\colon S \to \mathbf{R}_u \times \mathbf{R}_s$. Barbot and Fenley showed independently that this map is bijective if and only if the Anosov flow is the suspension of some Anosov diffeomorphism of the 2-torus. In all other cases, they showed that the image of (p_u, p_s) is an open strip in $\mathbf{R}_u \times \mathbf{R}_s$ of the form $\{(x,y) \mid h_-(x) < y < h_+(x)\}$ where h_- and h_+ are some homeomorphisms from $\mathbf{R}_u$ to $\mathbf{R}_s$. Now, observe that the fundamental group Γ of the manifold M acts on all these objects so that we get in particular actions of Γ on $\mathbf{R}_u$ and $\mathbf{R}_s$ which are conjugate by h_u and h_s. Denote by τ the composition $h_u h_s^{-1}$: this is a homeomorphism of $\mathbf{R}_u$ which acts freely so that we can define a circle $\mathbf{S}^1_u$ by taking the quotient of $\mathbf{R}_u$ by the action of τ. Note that the action of Γ on $\mathbf{R}_u$ obviously commutes with τ so that we get an action of Γ on $\mathbf{S}^1_u$. In case we start with the geodesic flow of a negatively curved surface Σ, the fundamental group Γ is a central extension of the fundamental group $\pi_1(\Sigma)$ by $\mathbf{Z}$. The action of Γ that we get on $\mathbf{S}^1_u$ is not faithful: the center $\mathbf{Z}$ acts trivially and the induced action of $\pi_1(\Sigma)$ on the circle is of course the familiar projective action. If the $\mathbf{R}$-covered Anosov flow is not the geodesic flow (up to a finite cover), the action of Γ on $\mathbf{S}^1_u$ is faithful. For instance, we get in this way some examples of faithful actions of the fundamental group of some hyperbolic 3-manifolds on the circle.

REFERENCES

[1] ARNOLD, V. *Chapitres supplémentaires de la théorie des équations différentielles ordinaires.* Reprint of the 1980 edition. "Mir", Moscow, 1984.

[2] BANYAGA, A. The structure of classical diffeomorphism groups. *Mathematics and its Applications 400.* Kluwer Academic Publishers Group, Dordrecht, 1997.

[3] BARBOT, T. Caractérisation des flots d'Anosov en dimension 3 par leurs feuilletages faibles. *Ergodic Theory Dynam. Systems 15* (1995), 247–270.

[4] BARGE, J. and E. GHYS. Cocycles d'Euler et de Maslov. *Math. Ann. 294* (1992), 235–265.

[5] BAVARD, C. Longueur stable des commutateurs. *L'Enseign. Math. (2) 37* (1991), 109–150.

[6] BEKLARYAN, L. A. About analogues of Tits' alternative for the homeomorphism group of the circle and real line. Preprint 2001 (in Russian).

[7] BOREL, A. and N. WALLACH. Continuous cohomology, discrete subgroups, and representations of reductive groups. *Annals of Mathematical Studies 94* (1980).

[8] BRIN, M. The chameleon groups of Richard J. Thompson: automorphisms and dynamics. *Inst. Hautes Études Sci. Publ. Math. 84* (1996), 5–33 (1997).

[9] BRIN, M. and M. GUZMÁN. Automorphisms of generalized Thompson groups. *J. Algebra 203* (1998), 285–348.

[10] BRIN, M. and C. SQUIER. Groups of piecewise linear homeomorphisms of the real line. *Invent. Math. 79* (1985), 485–498.

[11] BROWN, K. *Cohomology of Groups.* Corrected reprint of the 1982 original. Graduate Texts in Mathematics *87*. Springer-Verlag, New York, 1994.

[12] BURGER, M. and N. MONOD. Bounded cohomology of lattices in higher rank Lie groups. *J. Eur. Math. Soc. (JEMS) 1* (1999), 199–235.

[13] CANNON, J., W. FLOYD and W. PARRY. Introductory notes on Richard Thompson's groups. *L'Enseign. Math. (2) 42* (1996), 215–256.

[14] CARTER, D. and G. KELLER. Bounded elementary generation of $\mathrm{SL}(n,\mathcal{O})$. *Amer. J. Math. 105* (1983), 673–687.

[15] CASSON, A. and D. JUNGREIS. Convergence groups and Seifert fibered 3-manifolds. *Invent. Math. 118* (1994), 441–456.

[16] DEHORNOY, P. A fast method for comparing braids. *Adv. Math. 125* (1997), 200–235.

[17] EISENBUD, D., U. HIRSCH and W. NEUMANN. Transverse foliations of Seifert bundles and self-homeomorphisms of the circle. *Comment. Math. Helv. 56* (1981), 638–660.