2. Basic properties of simple triangle surfaces

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **09.08.2024**

THEOREM B. *For every* $k \geq 2$ *and* $g = \frac{k}{2}(k+1)$ *the Teichmüller space* $\mathcal{T}_{g,0}$ *can be parametrized by the length functions of* $6g+3$ *free homotopy classes contained in the orbit of a fixed class under a maximal finite subgroup* G *of* $Map(g,0)$. *The group* G *is a semidirect product of a cyclic group of order* $2g+1$ *and a cyclic group of order* 3.

We refer to [S2] for a discussion of other interesting parametrizations of $\mathcal{T}_{g,0}$.

The structure of this note is as follows. In Section 2 we look at simple triangle surfaces with additional symmetries. In Section 3 we give a combinatorial description of a family of curves which contains the systoles of every simple triangle surface. Length estimates in Section 4 lead to a complete description of the systoles of a simple triangle surface. This is used in Section 5 to show our theorems.

As a notational convention, we number the vertices of a fundamental $2p$-gon Ω counter-clockwise in consecutive order and we number and orient the edges of Ω in such a way that the edge i as an oriented arc joins the vertex $i-1$ to the vertex i. Moreover we write simply $\mathcal{T}_g$ for the Teichmüller space of marked hyperbolic structures on a closed surface of genus g.

2. BASIC PROPERTIES OF SIMPLE TRIANGLE SURFACES

Let $g \geq 2$ and let $p = 2g+1$. There is up to isometry a unique $2p$-gon Ω in the hyperbolic plane $\mathbf{H}^2$ with geodesic sides of equal length and with angles $2\pi/p$. In the introduction we called Ω a *fundamental* $2p$*-gon.* The *center* of Ω is the unique point $z \in \Omega$ which has the same distance to each of the vertices. A fundamental $2p$-gon admits a cyclic group Γ of isometries whose elements rotate Ω about the center with a rotation angle which is a multiple of $2\pi/p$. We view Γ as a cyclic group of isometries of the whole hyperbolic plane $\mathbf{H}^2$.

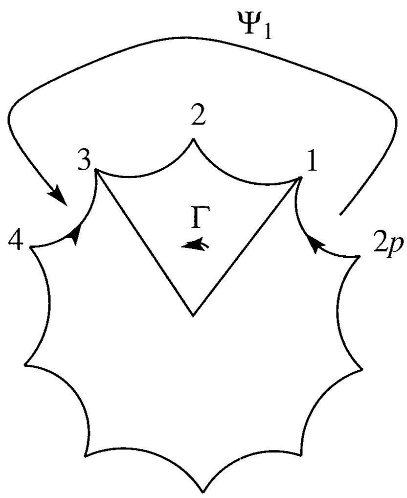
We call a closed hyperbolic surface S of genus g a *simple triangle surface* if $S = \mathbf{H}^2/G$ where G is a discrete torsion free group $G \subset PSL(2,\mathbf{R})$ of isometries of $\mathbf{H}^2$ which is normalized by the group Γ and which admits Ω as a fundamental polygon (see [M] for basic informations on fundamental polygons). The group G then acts as a group of side pairing transformations for the polygon Ω. This means that for each side a of Ω there is an isometry $\Psi \in G$ which maps a to a second side $\Psi(a) \neq a$ of Ω in such a way that $\Psi(\Omega) \cap \Omega = \Psi a$.

Our first observation is that simple triangle surfaces exist for every genus $g \geq 2$.

LEMMA 2.1. *For every $g \geq 2$ there is a simple triangle surface of genus g.*

Proof. Let $p \geq 5$ be an odd number and let Ω be a fundamental $2p$-gon with center $0 \in \mathbf{H}^2$. We have to show that there is a discrete subgroup G of $PSL(2, \mathbf{R})$ which is normalized by Γ and which admits Ω as a fundamental polygon.

Choose a number $k \in \{2, \ldots, p-1\}$ and define a family $\{\Psi_1, \ldots, \Psi_p\}$ of isometries of $\mathbf{H}^2$ by requiring that Ψ_j maps the (oriented) edge with odd number $2j+1$ orientation reversing onto the (oriented) edge $2j+2k$ in such a way that $\Psi_j(\Omega) \cap \Omega$ is just the edge $2j+2k$. Then necessarily the vertex $2j$ is mapped to the vertex $2j+2k$, and the vertex $2j+1$ is mapped to the vertex $2j+2k-1$. We claim that these isometries $\{\Psi_1, \ldots, \Psi_p\}$ generate a discrete subgroup of $PSL(2, \mathbf{R})$ with fundamental domain Ω if and only if k and $k-1$ are prime to p.



To see this let G be the subgroup of $PSL(2, \mathbf{R})$ generated by $\Psi_1, \ldots, \Psi_p$ and assume that G is discrete and torsion free, with fundamental polygon Ω. By the choice of $\Psi_1, \ldots, \Psi_p$, the G-orbit of an even (or odd) vertex of Ω intersects Ω only in the set of even (or odd) vertices. Different such vertex cycles project to different points on the surface $S = \mathbf{H}^2/G$. If $m \geq 2$ is the number of points in the vertex cycle of the vertex a, then a neighborhood of the projection $\bar{a}$ of a to S consists of $2m$ equilateral hyperbolic triangles with angle π/p which contain $\bar{a}$ as one of their vertices. Since S is a smooth hyperbolic surface, the angles at $\bar{a}$ of these triangles must add up to 2π. This means that there are precisely 2 vertex cycles for the action of G, each containing only even or only odd vertices. By the definition of G this is the case if and only if the number $k \in \{2, \dots, p-1\}$ is prime to p and $k-1$ is prime to p as well. Such a group G is then normalized by the group Γ of rotations of Ω with rotation angle a multiple of 2π.

The same argument also shows that for $k \in \{2, \dots, p-1\}$ which is prime to p and such that $k-1$ is prime to p as well the group G induces a simple triangle surface of genus g. Since $p = 2g+1$ is odd we can always choose $k = 2$ to obtain an example. $\square$

In the above proof we observed that we obtain a simple triangle surface from a fundamental $2p$-gon Ω by identifying the edge 1 with the edge $2k$ for some $k \in \{2, \dots, p-1\}$ if and only if k and $k-1$ are prime to p. We denote by $S(p;k)$ the surface obtained in this way. For fixed $p \geq 5$ this defines a finite non-empty collection of simple triangle surfaces of genus $\frac{1}{2}p - 1$ indexed by the set of all numbers $k \in \{2, \dots, p-1\}$ which are prime to p and such that $k-1$ is prime to p as well. However these surfaces are not necessarily distinct as hyperbolic surfaces. For example, via exchanging the roles of the even and odd vertices of our fundamental $2p$-gon Ω we observe that the surface $S(p;k)$ is isometric to the surface $S(p;p-k+1)$. Thus we may restrict our attention to the case that $k \leq \frac{1}{2}(p+1)$. In the sequel we sometimes identify the surfaces $S(p;k)$ and $S(p;p-k+1)$ without further comment.

Let again Γ be the group of rotations of Ω which descends to a group of isometries on a simple triangle surface S of genus g. The natural Γ-invariant triangulation of Ω into $2p$ equilateral triangles with angle π/p projects to the Γ-invariant canonical triangulation whose 3 vertices $0, A, B$ are just the fixed points for the action of Γ. The quotient S/Γ of S under Γ is a topological 2-sphere. The hyperbolic metric on S projects to a hyperbolic metric on S/Γ with 3 singular points $\widehat{A}, \widehat{B}, \widehat{0}$ which are the projections of the vertices $A, B, 0$ of the canonical triangulation of S. With this metric, S/Γ is isometric to two equilateral hyperbolic triangles with angle π/p glued at their boundaries. This observation is used in the proof of the following.

LEMMA 2.2.

1) *Let $p \geq 5$ be an odd number and let $k, m \in \{2, \dots, p-1\}$ be numbers which are prime to p and such that $k-1, m-1$ are prime to p as well. If either $(k-1)m+1 \equiv 0 \mod p$ or $(m-1)k+1 \equiv 0 \mod p$ then the surfaces $S(p;k)$ and $S(p;m)$ are isometric.*

2) *A simple triangle surface* S *with basic group* Γ *of isometries admits a nontrivial group* $\Sigma \not\subset \Gamma$ *of orientation preserving isometries which normalizes* Γ *if and only if one of the following holds.*

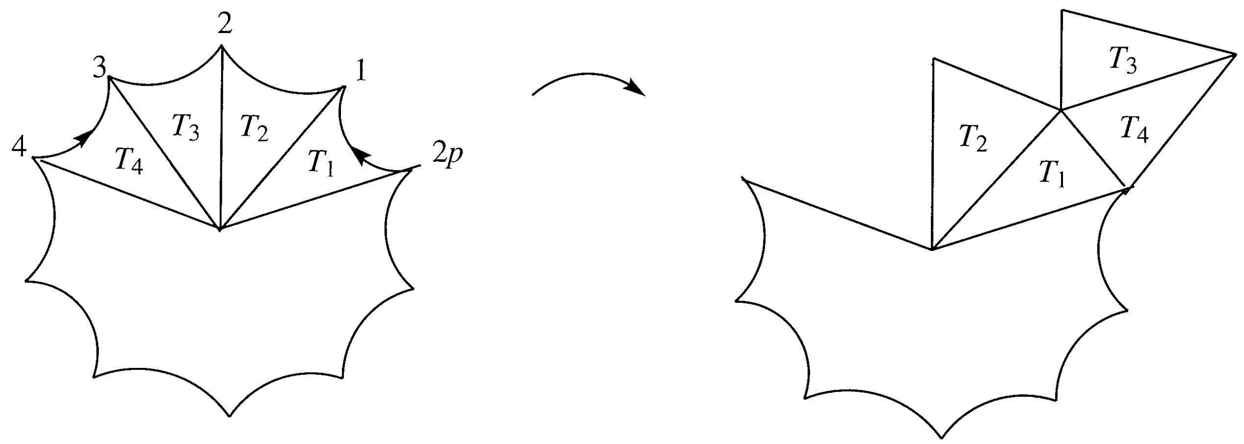
i) $S = S(p;k)$ *for some* $k \geq 2$ *and a divisor* $p \geq k+1$ *of* $k(k-1)+1$. *The group* Σ *is then cyclic of order* 3.

ii) $S = S(p;2)$ *and the group* Σ *is cyclic of order* 2 *and generated by a hyperelliptic involution.*

Proof. Let $p \geq 5$ and let $k \leq p-1$ be such that $k-1$ and k are prime to p. Let Ω be a fundamental $2p$-gon and let $0, A, B$ be the vertices of the canonical triangulation of S. We assume that 0 is the projection of the center of Ω and A is the projection of the odd vertices of the boundary of Ω.

As in the introduction we number the $2p$ edges of Ω in counterclockwise order in such a way that the edge i is adjacent to the vertices $i-1$ and i. Let $T_i \subset S$ be the projection of the triangle in Ω with one vertex at the center of Ω and with the edge i of Ω as the opposite side. The triangles $T_1, \dots, T_{2p}$ are arranged in counterclockwise order around the vertex 0.

There is a different representation of S as a quotient of Ω under a group of side pairing transformations in such a way that the center of Ω projects to the vertex A of the canonical triangulation. Namely, if we cut S open along the geodesic arcs connecting the vertices 0 and B, then the result is a fundamental $2p$-gon which consists again of the triangles $T_1, \dots, T_{2p}$. The arrangement of these triangles around the vertex A is given by a permutation σ of $\{1, \dots, 2p\}$ with $\sigma(1) = 1$, i.e. the counterclockwise order of the triangles around the vertex A is $T_{\sigma(1)}, \dots, T_{\sigma(2p)}$. The parity of $\sigma(i)$ coincides with the parity of i. Moreover for every $i \in \{1, \dots, p\}$ we have $\sigma(2i) = \sigma(2i+1)+1$ mod $2p$.



The side pairings of Ω which define S in such a way that the center of Ω projects to 0 glue the edge $2k$ to the edge 1 and therefore we have

$\sigma(2) = 2k$ and $\sigma(3) = 2k-1$. The basic group Γ of isometries of S permutes the triangles T_i and fixes the vertex A. This implies that σ normalizes the group of permutations of $\{1,\dots,2p\}$ generated by the permutation $\tau(i) = i+2 \mod 2p$ and hence necessarily $\sigma(2i) = 2i(k-1)+2$.

To obtain our surface S we have to identify the edge $2i-1$ with the edge $2im$ for some $m \in \{2,\dots,p-1\}$ with an orientation reversing isometry. The number m is uniquely determined if we require in addition that the triangles adjacent to odd edges of Ω project once again to the triangles T_{2i-1} $(i = 1,\dots,p)$ of the canonical triangulation.

Comparing the arrangement of triangles around 0 and A we conclude that $\sigma(2m) = 2p$. Together with the above this shows that $2m(k-1)+2 \equiv 0 \mod 2p$ or, equivalently, $m(k-1)+1 \equiv 0 \mod p$. In other words, if $m,k \geq 2$ are such that $m(k-1)+1 \equiv 0 \mod p$ then the surfaces $S(p;k)$ and $S(p;m)$ are isometric. This shows the first part of the lemma.

To show the second part of our lemma let S be a simple triangle surface which admits a non-trivial group Σ of orientation preserving isometries normalizing the basic group Γ. Then the action of Σ on S descends to an isometric action on the sphere S/Γ. The sphere S/Γ consists of two equilateral triangles with angle π/p glued at their boundaries. One of these triangles is the projection of the odd triangles of the canonical triangulation of S, the other one is the projection of the even triangles.

Every isometry of S/Γ has to preserve the singular set $\{\widehat{A},\widehat{B},\widehat{0}\} \subset S/\Gamma$ of ramification points which consists of the vertices of the two triangles forming S/Γ. The only nontrivial isometry of S/Γ which fixes each of the ramification points $\widehat{0},\widehat{A},\widehat{B}$ is the orientation reversing reflection which exchanges the two triangles forming S/Γ. By assumption the elements of Σ preserve the orientation of S and hence of S/Γ, and therefore there are two possibilities:

1) Σ contains an element Ψ which permutes cyclicly the singular points $\widehat{A},\widehat{B},\widehat{0}$ of S/Γ and preserves each of the two triangles which form S/Γ.

2) Σ fixes one singular point of S/Γ, permutes the two other ones and exchanges the two triangles which form S/Γ.

Assume that $S = S(p;k)$ admits an isometry Ψ as in 1) above. Then Ψ permutes the triangles of the canonical triangulation, but preserves their parity. If we cut $S = S(p;k)$ open along those edges of the triangles of the canonical triangulation which connect the vertices A and B, then the result is the fundamental $2p$-gon Ω and we obtain our surface from Ω by a side pairing which identifies the edges 1 and $2k$. Since Ψ is an isometry of S

which preserves the canonical triangulation, if we cut S open along the edges connecting the vertices $\Psi(A)$ and $\Psi(B)$ then the result is again the polygon Ω, and once again we obtain S from Ω by identifying the edges 1 and $2k$. This together with the above consideration shows that $k(k-1)+1 \equiv 0 \mod p$ and therefore p divides $k(k-1)+1$.

Assume now that S admits an isometry Ψ as in 2) above. Then Ψ permutes the triangles of the canonical triangulation and changes their parity with respect to a given counter clockwise numbering around a given vertex. Let $m \leq p-1$ be such that $k(m-1)+1 \equiv 0 \mod p$. The above considerations imply that necessarily $k = p - m + 1$ and hence $(m-1)^2 \equiv 1 \mod p$ or equivalently $m(m-2) \equiv 0 \mod p$. Since $m \geq 1$ is prime to p we conclude that either $m = 2$ or that p divides $m-2$. But $m \leq p-1$ and therefore only the case $m = 2$ is possible.

We are left with showing that the isometry Ψ is a hyperelliptic involution. For this notice that every fixed point of Ψ projects to a fixed point for the induced isometry $\widehat{\Psi}$ of S/Γ. The map $\widehat{\Psi}$ has precisely two fixed points: A singular point $\widehat{0}$ of S/Γ and the midpoint y of the geodesic arc connecting the two other singular points.

There are exactly $p = 2g+1$ preimages of y in S. Since $\Psi^2 = Id$ and since Ψ normalizes Γ, either every preimage or no preimage is fixed by Ψ. The Riemann Hurwitz-formula [F] shows that the second case is impossible. Thus Ψ has exactly $p+1 = 2g+2$ fixed points and is a hyperelliptic involution. □

COROLLARY 2.3. *For every $g \geq 2$ there is a hyperelliptic surface of genus g whose full automorphism group is the direct product of a cyclic group of order $2g+1$ and a cyclic group of order 2 generated by a hyperelliptic involution.*

Proof. We showed in Lemma 2.1 that for each $g \geq 2$ there is a simple triangle surface $S(2g+1;2)$. By Lemma 2.2 and its proof, this surface is hyperelliptic and its isometry group is a stated in the corollary. □

REMARK. There are surfaces $S(p;k)$ for $p \notin \{\ell(\ell-1)+1 \mid \ell \geq 2\}$ which admit a cyclic group Σ of isometries of order 3 contained in the normalizer of the basic group Γ. The simplest surface of this kind is the surface $S(19;8)$ of genus $g = 9$.