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THE POSITIVE CONE OF SPHERES AND SOME PRODUCTS OF SPHERES

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ABSTRACT. Motivated by Elliott's K-theoretic classification of C^*-algebras of type AF, we compute the positive cone of the K-theory of some spaces. These include the spheres, the products of an odd-dimensional sphere by a sphere, the products of the 2-sphere by a sphere, and of the products $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This amounts to computing the geometric dimension of stable classes of complex vector bundles over these spaces. We establish a few general properties of the positive cone and of approximations to it, the γ-cone and the c-cone. We also get information on the Whitehead product structure in the homotopy groups of $BU(n)$. Moreover, we prove a "doubling formula" for Stirling numbers of the second kind.

1. INTRODUCTION

Let $\mathcal{G}(S)$ be the Grothendieck group completion of an abelian semigroup S, and let $\theta \colon S \longrightarrow \mathcal{G}(S)$ be the corresponding universal homomorphism. The image of θ, denoted by $\mathcal{G}_+(S)$, is a sub-semigroup of $\mathcal{G}(S)$. If S has a zero, in other words if it is an abelian monoid, then $\mathcal{G}_+(S)$ induces a translation invariant preordering on $\mathcal{G}(S)$ (i.e. a reflexive and transitive relation, but not necessarily antisymmetric). The elements of $\mathcal{G}_+(S)$ are called *positive* and $\mathcal{G}_+(S)$ is called the *positive cone* (see [Ell] and [Bla1]). The pair $(\mathcal{G}(S), \mathcal{G}_+(S))$ is an isomorphism invariant of S, and a basic question is: to what extent does this invariant characterize the abelian semigroup S ?

The above notions are of interest in connection with the classification problem of C^*-algebras. For a unital C^*-algebra A, let $S = \mathrm{Proj}(A)$ be the abelian monoid of equivalence classes of projectors in the matrix algebra $\mathbf{M}_\infty(A)$. The K-theory of A, denoted by $K_0(A)$ or $K(A)$, is by definition

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the group $\mathcal{G}(\mathrm{Proj}(A))$. The positive cone in K-theory is $\mathcal{G}_+(\mathrm{Proj}(A))$ and it is denoted by $K_+(A)$. In [Ell], Elliott has put forward a program to classify a large class of unital C^*-algebras by invariants of a K-theoretic nature, such as $K(A)$, $K_+(A)$, [1] (the K-theory class of the unit), etc. (see also [Bla1]). For a compact Hausdorff space X, the algebra $C(X)$ of continuous complex valued functions on X is a unital C^*-algebra and its K-theory coincides with the topological K-theory $K^0(X)$ of the space X (according to the Swan-Serre theorem). In view of Elliott's program and to shed light on various conjectures, it is of great interest to determine for such spaces the positive cone $K_+(X) = \mathcal{G}_+(\mathrm{Proj}(C(X)))$. For any connected X, the preordering determined on $K^0(X)$ by the positive cone is an ordering, as is easily checked (see also p. 84 in [Rord]).

The problem of computing the positive cone of some spaces and in particular of spheres has been communicated to us by Alain Valette, after a question asked by G. A. Elliott in Oberwolfach.

These notes are organized as follows. In Section 2, we recall the basic facts from topological K-theory needed in the sequel. Among other things, we review γ-operations. The computation of these operations for even-dimensional spheres puts Stirling numbers of the second kind on stage. In Section 3, we define what we call the γ-cone and the c-cone (the latter is defined in terms of Chern classes), and we explain in what sense they are approximations of the positive cone. We illustrate by examples that the three notions of cones are different in general, although the γ-cone and the c-cone coincide for torsion-free spaces.

In Section 4, we compute the positive cones of the spheres, by using some standard homotopy theory. Section 5 is devoted to the naturality properties of the three cones. The positive cone of the products $S^n \times S^{2m-1}$ is computed in Section 6. The γ-cone of the products $S^{2n} \times S^{2m}$ is easily calculated in Section 7 by means of Chern classes. In that section, we also compute the positive cone of $S^2 \times S^{2n}$.

The Whitehead product structure on the homotopy of the classifying space $BU(n)$ is closely related to the problem of determining the positive cone of the product of two even-dimensional spheres, as is explained in Section 8. This allows us to improve slightly a result of Bott on this structure, and gives some precise information on the positive cone of such a product of spheres.

In Section 9, we perform the computation of the positive cones of $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and of $S^6 \times S^8$. This is achieved by using some well-known results on the homotopy groups of unitary groups. In Section 10, we show that for spaces "with only one high-dimensional cell" the γ-cone is "blind" in some sense to be made precise there.

Section 11 is devoted to explicitly computing the γ-operations for the products $S^{2n} \times S^{2m}$. As a consequence of these calculations, we establish a "doubling-formula" for Stirling numbers of the second kind. Moreover, we are led to conjecture that the same formula holds for Stirling numbers of the first kind. (This has now been proved by Al Lundell; see Theorem 11.2.)

2. Preliminaries

We start by reviewing some topological K-theory. Our basic references are the books by Atiyah [Atiyah] and by Husemoller [Huse].

Let X be a *connected finite* CW-complex. (We assume all spaces and maps to be pointed.) For each $n \geq 0$, let $\mathrm{Vect}_n(X)$ be the set of isomorphism classes of complex n-plane vector bundles over X, and $\mathrm{Vect}(X)$ their disjoint union. There are well-known bijections

$$\mathrm{Vect}_n(X) \approx [X, BU(n)] \qquad (n \geq 0)$$

where $BU(n)$ is the classifying space of the unitary group $U(n)$ and $[\,.\,,\,.\,]$ stands for the set of homotopy classes of maps. For an n-plane vector bundle ξ over X, i.e. $\xi \in \mathrm{Vect}_n(X)$, we write $\mathrm{rk}(\xi) = n$ (it is the *rank* of ξ). The direct sum (also called Whitney sum) and the tensor product of vector bundles endow $\mathrm{Vect}(X)$ with a semiring structure. The K-theory of X is the ring $K(X)$, also denoted by $K^0(X)$, obtained by applying the Grothendieck construction to $\mathrm{Vect}(X)$, i.e. $K(X) = \mathcal{G}(\mathrm{Vect}(X))$. An element of $K(X)$ is sometimes called a *virtual vector bundle*. There is a ring isomorphism

$$K(X) \cong [X, \mathbf{Z} \times BU],$$

where BU is the infinite Grassmannian, i.e. the direct limit of the classifying spaces $BU(n)$. We identify both rings from now on. There is a canonical splitting $K(X) = \mathbf{Z} \oplus [X, BU] = \mathbf{Z} \oplus \widetilde{K}(X)$, where $\widetilde{K}(X) = \widetilde{K}^0(X)$ is the subring of *stable classes* of vector bundles, and $n \in \mathbf{N} = \{0, 1, 2, \ldots\}$ is represented by the n-dimensional trivial vector bundle over X. Clearly, the Grothendieck construction gives rise to maps $\theta\colon \mathrm{Vect}(X) \longrightarrow K(X)$ and $\theta_n\colon \mathrm{Vect}_n(X) \longrightarrow n \times \widetilde{K}(X)$ (by restriction of θ).

DEFINITION 2.1.

i) The *positive cone* of X, denoted by $K_+(X)$, is the image of θ. An element $\xi \in K(X)$ is called *positive* if it lies in the positive cone.

ii) The *geometric dimension* of $x \in \widetilde{K}(X)$, denoted by g-dim(x), is the smallest integer n such that (n, x) lies in the image of θ_n, i.e. the least integer n such that the stable class x is represented by an n-dimensional vector bundle.

Since θ is a semiring homomorphism, it is clear that $K_+(X)$ is a subsemiring of $K(X)$. Notice that it is equivalent to determine the positive cone or the map g-dim: $\widetilde{K}(X) \longrightarrow \mathbf{Z}$; in fact, we have

$$K_+(X) = \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid n \geq \text{g-dim}(x)\}.$$

Let us also notice that an element $x \in \widetilde{K}(X)$, considered as a homotopy class of maps $X \longrightarrow BU$, has geometric dimension $\leq n$ if and only if x has a lifting $x_n \colon X \longrightarrow BU(n)$, i.e.

$$\begin{tikzcd} & BU(n) \arrow[d, "i_n"] \\ X \arrow[ur, "x_n"] \arrow[r, "x"] & BU \end{tikzcd} \qquad i_n \circ x_n = x$$

(Here, we identify a map with the homotopy class it represents.) Recall that i_n is a fibration with fibre $U/U(n)$, where $U = \varinjlim U(n)$ is the infinite unitary group (and BU is really its classifying space). The image of θ_n is equal to the image of the composition

$$[X, BU(n)] \xrightarrow{(i_n)_*} [X, BU] \hookrightarrow n \times [X, BU], \qquad y \longmapsto (n, (i_n)_*(y)).$$

We write $K^*(X) = K^0(X) \oplus K^1(X)$, where the K^1-group is defined by

$$K^1(X) := [X, U].$$

For a pair of connected finite CW-complexes (X, Y), there is the famous six-term exact sequence:

$$\begin{tikzcd} \widetilde{K}^0(X/Y) \arrow[r, "q^*"] & \widetilde{K}^0(X) \arrow[r, "i^*"] & \widetilde{K}^0(Y) \arrow[d] \\ K^1(Y) \arrow[u] & K^1(X) \arrow[l, "i^*"'] & K^1(X/Y), \arrow[l, "q^*"'] \end{tikzcd}$$

where $i \colon Y \hookrightarrow X$ is the inclusion and $q \colon X \twoheadrightarrow X/Y$ is the quotient map.

The n-th exterior power operation for complex vector spaces induces an operation on vector bundles denoted by $\xi \longmapsto \lambda^n\xi$, and endows $K(X)$ with a natural λ-ring structure. For $\xi \in K(X)$, one defines

$$\lambda_t(\xi) := \sum_{n\geq 0}(\lambda^n\xi)\cdot t^n \in K(X)[[t]]$$

(the latter being the ring of formal power series with coefficients in $K(X)$). The function λ_t is exponential, i.e. $\lambda_t(\xi+\eta) = \lambda_t(\xi)\cdot\lambda_t(\eta)$. Associated to the *$\lambda$-operations* are the *$\gamma$-operations* or *Grothendieck operations* $\gamma^n(\xi)$, which are defined by their generating series as follows:

$$\sum_{n\geq 0}\gamma^n(\xi)\cdot t^n = \gamma_t(\xi) := \lambda_{t/1-t}(\xi)\,.$$

In particular, $\gamma^0(\xi) = 1$ and $\gamma^1(\xi) = \xi$. Again, the function γ_t is exponential, which implies that

$$\gamma^n(\xi+\eta) = \sum_{k=0}^{n}\gamma^k(\xi)\cdot\gamma^{n-k}(\eta)\,.$$

The importance of the γ-operations in our context is illustrated by the following fact (see [Atiyah], Proposition 3.1.1):

Let $x \in \widetilde{K}(X)$; if $\text{g-dim}(x) \leq n$, then $\gamma^k(x) = 0$ for $k > n$.

(Assume that $(n, x) \in \mathbf{Z}\oplus\widetilde{K}(X)$ is represented by an n-dimensional vector bundle ξ. Then $\lambda_s(\xi)$ is a polynomial of degree n in s. By the exponential property, $\lambda_s(\xi) = \lambda_s(1)^n\cdot\lambda_s(x) = (1+s)^n\cdot\lambda_s(x)$. Letting $s = t/(1-t)$, we see that $\gamma_t(x) = \lambda_{t/1-t}(x) = (1-t)^n\lambda_{t/1-t}(\xi)$ is a polynomial of degree $\leq n$ in t.)

The representable K-theory of $BU(n)$, i.e. $[BU(n), \mathbf{Z}\times BU]$, is

$$K(BU(n)) = \mathbf{Z}[[\widetilde{\gamma}^1, \ldots, \widetilde{\gamma}^n]]\,,$$

where $\widetilde{\gamma}^k = \gamma^k(\widetilde{\rho}_n)$, for $1 \leq k \leq n$, $\widetilde{\rho}_n$ being the stable class of the universal n-plane bundle ρ_n over $BU(n)$. Note that $\gamma^k(\widetilde{\rho}_n) = 0$, for all $k > n$, and that the map j^*, induced by $j\colon BU(n) \longrightarrow BU(n+l)$ in K-theory, takes $\widetilde{\rho}_{n+l}$ to $\widetilde{\rho}_n$, for any $l \geq 0$.

For a complex vector bundle ξ over X, the *n-th Chern class* $c_n(\xi)$ is a $2n$-dimensional integral cohomology class of X, i.e. $c_n(\xi) \in H^{2n}(X;\mathbf{Z})$. One has $c_0(\xi) = 1$, and the element $c(\xi) = \sum_{n\geq 0}c_n(\xi) \in H^*(X;\mathbf{Z})$, called the *total Chern class*, is exponential, i.e. it satisfies

$$c(\xi+\eta) = c(\xi)\cdot c(\eta)\,.$$

The basic properties of Chern classes (see [Huse]) imply the following facts:

i) Two stably equivalent bundles over X have the same Chern classes. In particular, for an element $x \in \widetilde{K}(X)$, the n-th Chern class $c_n(x) \in H^{2n}(X; \mathbf{Z})$ is well-defined.

ii) If $n > \mathrm{rk}(\xi)$, then $c_n(\xi) = 0$.

iii) Let $x \in \widetilde{K}(X)$; if g-dim$(x) \leq n$, then $c_k(x) = 0$ for $k > n$.

Let us also formally define the polynomial

$$c_\xi(t) := \sum_{n \geq 0} c_n(\xi) \cdot t^n \in H^*(X; \mathbf{Z})[t],$$

which, by ii) above, is indeed a polynomial. It is also exponential.

A central feature of Chern classes is that the cohomology ring of $BU(n)$ is given by

$$H^*(BU(n); \mathbf{Z}) = \mathbf{Z}[\widetilde{c}_1, \ldots, \widetilde{c}_n],$$

where $\widetilde{c}_k = c_k(\widetilde{\rho}_n)$, for $1 \leq k \leq n$. Moreover, $c_k(\widetilde{\rho}_n) = 0$, for any $k > n$. On the combinatorial point of view, for any $n \geq 0$, the Grassmannians $BU(n)$ and BU admit CW-decompositions with the same $(2n+1)$-skeleton, in other words, such that $BU(n)^{[2n+1]} = BU^{[2n+1]}$. (This can be proved by adapting Section 6 of [MilSt] to the complex case.)

The Chern character ch is a multiplicative natural transformation from K-theory to rational cohomology

$$ch\colon K(X) \longrightarrow H^{ev}(X; \mathbf{Q}) = \bigoplus_{q \geq 0} H^{2q}(X; \mathbf{Q}), \quad \xi \longmapsto ch(\xi) = \sum_{q \geq 0} ch_{2q}(\xi),$$

where $ch_{2q}(\xi) \in H^{2q}(X; \mathbf{Q})$ (X being a connected *finite* CW-complex). It relates γ-operations and Chern classes as given in the following well-known proposition. Before stating it, we introduce some notation. For $x \in \widetilde{K}(X)$, we let $\bar{c}_j(x)$ be the image of $c_j(x)$ under the coefficient homomorphism $H^{2j}(X; \mathbf{Z}) \longrightarrow H^{2j}(X; \mathbf{Q})$, and let $I_x(\bar{c}_k, \ldots, \bar{c}_n)$ be the ideal in $H^{ev}(X; \mathbf{Q})$ generated by $\bar{c}_k(x), \ldots, \bar{c}_n(x)$, where $x \in \widetilde{K}(X)$ and $n \geq k$.

PROPOSITION 2.2. *Let X be a connected finite CW-complex of dimension $\leq 2n$, $x \in \widetilde{K}(X)$, and $k \leq n$. One then has*

$$ch(\gamma^k(x)) = \bar{c}_k(x) + P_{k+1}(\bar{c}_1(x), \ldots, \bar{c}_n(x)),$$

where P_{k+1} is a polynomial in $\bar{c}_1(x), \ldots, \bar{c}_n(x)$ contained in the ideal

$$I_x(\bar{c}_k, \ldots, \bar{c}_n) \cap \Big(\bigoplus_{q \geq k+1} H^{2q}(X; \mathbf{Q}) \Big).$$

In particular,

$$ch(\gamma^n(x)) = \bar{c}_n(x) \in H^{2n}(X;\, \mathbf{Q})\,.$$

Proof. For a line bundle $\eta = 1+y$, one has $ch(\gamma_t(y)) = 1 + (e^{c_1(y)} - 1)\cdot t$ in $H^{ev}(X;\, \mathbf{Q})[[t]]$, therefore, the result follows readily for the Whitney sum of n universal line bundles over $\mathbf{C}P^\infty \times \ldots \times \mathbf{C}P^\infty$ (n factors) and hence for any element x represented by a Whitney sum of line bundles. The general case is obtained by invoking the splitting principle. □

(Proposition 2.2 shows that $\gamma^k(x)$ is of filtration $\geq k$, in the terminology of Atiyah and Hirzebruch [AtHi].)

Recall that a finite CW-complex is called *torsion-free* if its integral homology (or equivalently integral cohomology) contains no torsion. The Chern classes give some "sharp information" on the geometric dimension, as the next fundamental theorem shows.

THEOREM 2.3. *Consider a connected finite CW-complex X of dimension $\leq 2n$, and $x \in \widetilde{K}(X)$. Then*

$$\text{g-dim}(x) < n \iff c_n(x) = 0\,.$$

If moreover X is torsion-free, then this is also equivalent to $\gamma^n(x) = 0$.

Proof. If $\text{g-dim}(x) < n$, as already mentioned, $c_n(x) = 0$. The converse is a consequence of Theorem 41.5 on page 210 of [Steen] (the Chern classes of a vector bundle are the same as the Chern classes of the associated spherical bundle as defined in [Steen]). The last statement follows from Proposition 2.2 and injectivity of the Chern character for a torsion-free space. □

Let us recall the K-theory of the spheres:

$$K^0(S^{2n}) \cong \mathbf{Z} \oplus \mathbf{Z} \qquad K^0(S^{2n+1}) \cong \mathbf{Z}$$
$$K^1(S^{2n}) = 0 \qquad K^1(S^{2n+1}) \cong \mathbf{Z}\,.$$

(For technical reasons, we will always implicitly exclude the 0-sphere.) The multiplicative structure on $\widetilde{K}(S^{2n}) = \mathbf{Z}\cdot x_{2n}$ is given by $x_{2n}^2 = 0$. The γ-operations and the Chern classes are as given in the next proposition.

PROPOSITION 2.4. *Let x_{2n} be a generator of $\widetilde{K}(S^{2n}) \cong \mathbf{Z}$. Then*

i) $\gamma^k(x_{2n}) = (-1)^{k-1}(k-1)!\, S(n, k) \cdot x_{2n}$, *where $S(n,k)$ is a Stirling number of the second kind.*

ii) $c_n(x_{2n}) = (-1)^{n-1}(n-1)! \cdot a_{2n}$, *where a_{2n} is a suitable generator of the cohomology group $H^{2n}(S^{2n}; \mathbf{Z}) \cong \mathbf{Z}$.*

Proof. It is well-known that $\lambda^k(x_{2n}) = (-1)^{k+1}k^{n-1} \cdot x_{2n}$, for $k \geq 1$ (see Proposition 2.5 and Theorem 11.2 in Chapter 13 of [Huse]). We thus get

$$\begin{aligned}
\gamma_t(x_{2n}) = \lambda_{t/1-t}(x_{2n}) &= 1 + \Big(\sum_{k\geq 1}(-1)^{k+1}k^{n-1}t^k(1-t)^{-k}\Big) \cdot x_{2n} \\
&= 1 - \Big(\sum_{k\geq 1}\sum_{j\geq 0}(-1)^{k+j}\tbinom{-k}{j}k^{n-1} \cdot t^{k+j}\Big) \cdot x_{2n} \\
&= 1 - \Big(\sum_{k\geq 1}\sum_{j\geq 0}(-1)^{k}\tbinom{k+j}{k}\tfrac{k^n}{k+j} \cdot t^{k+j}\Big) \cdot x_{2n} \\
&\overset{(*)}{=} 1 - \Big(\sum_{m\geq 1}\Big(\sum_{k=1}^{m}(-1)^{k}\tbinom{m}{k}\tfrac{k^n}{m}\Big) \cdot t^{m}\Big) \cdot x_{2n} \\
&= 1 + \Big(\sum_{m\geq 1}(-1)^{m-1}(m-1)!\, S(n, m) \cdot t^{m}\Big) \cdot x_{2n}\,,
\end{aligned}$$

since $S(n, m) = \sum_{k=1}^{m}(-1)^{m-k}\binom{m}{k}\frac{k^n}{m!}$ (see (6.19) on page 251 of [GKP]), hence the first formula. (The equality $(*)$ is obtained by substituting $m := k+j$.)

The second formula follows from Theorem 9.6 and Corollary 9.8 (and its proof) in Chapter 20 of [Huse]. $\square$

Let us finally state a lemma relating γ-operations and Chern classes. We shall need it further on.

LEMMA 2.5. *Let Y be a connected CW-complex (possibly infinite). Then, for an element $x \in \widetilde{K}(Y) = [Y, BU]$, one has*

$$c_n(\gamma^n(x)) = (-1)^{n-1}(n-1)!\, c_n(x) \in H^{2n}(Y; \mathbf{Z})\,.$$

Proof. Let $i\colon BU(n-1) \longrightarrow BU$ be the canonical map, and

$$i^*: H^*(BU; \mathbf{Z}) = \mathbf{Z}[\widetilde{c}_1, \widetilde{c}_2, \ldots] \longrightarrow \mathbf{Z}[\widetilde{c}_1, \ldots, \widetilde{c}_{n-1}] = H^*(BU(n-1); \mathbf{Z})$$

the induced map. Since $c_n(\widetilde{\gamma}^n) \in \mathrm{Ker}(i^*) \cap H^{2n}(BU; \mathbf{Z}) = \mathbf{Z} \cdot \widetilde{c}_n$, there exists an integer q_n such that $c_n(\widetilde{\gamma}^n) = q_n \cdot \widetilde{c}_n$. Recalling that $S(n, n) = 1$, an easy computation (based on Proposition 2.4) for the sphere S^{2n} shows that one has $q_n = (-1)^{n-1}(n-1)!$, as claimed. $\square$

3. THE γ-CONE AND THE c-CONE

In general, the problem of computing the geometric dimension of vector bundles is very complicated, as is any general lifting problem in homotopy theory. So, the same is true for the positive cone. That is why we now introduce what we call the γ-cone and the c-cone. They are supposed to be easier to compute and might be good approximations to the positive cone. As we will see, these two cones coincide for torsion-free spaces.

DEFINITION 3.1.
i) The *γ-cone* of X is defined by

$$K_\gamma(X) := \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid \gamma^k(x) = 0 \ \text{ for all } \ k > n\}.$$

The *γ-dimension* of a class $x \in \widetilde{K}(X)$, denoted by γ-dim(x), is the least integer n such that $\gamma^k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $\gamma_t(x)$.

ii) The *c-cone* of X is defined by

$$K_c(X) := \{(n, x) \in \mathbf{Z} \oplus \widetilde{K}(X) \mid c_k(x) = 0 \ \text{ for all } \ k > n\}.$$

The *c-dimension* of a class $x \in \widetilde{K}(X)$, denoted by c-dim(x), is the least integer n such that $c_k(x) = 0$ for all $k > n$, in other words, it is the degree (in the variable t) of the polynomial $c_x(t)$.

Let us point out that the "lower boundary" of the positive cone $K_+(X)$, as a subset of $\widetilde{K}(X) \oplus \mathbf{Z}$, coincides with the graph of the geometric dimension function g-dim: $\widetilde{K}(X) \longrightarrow \mathbf{Z}$ (the positive elements consisting exactly of the boundary and the points located above it). The analogous statements hold for the γ-cone and the c-cone with respect to the corresponding dimension function.

The following results on these objects follow readily from our preliminaries on K-theory.

PROPOSITION 3.2. *Let X be a connected finite CW-complex. Then*

i) $\text{g-dim}(x) \leq \dim(X)/2$, *for any* $x \in \widetilde{K}(X)$;

ii) $\gamma\text{-dim}(x) \leq \text{g-dim}(x)$, *for any* $x \in \widetilde{K}(X)$;

iii) $K_+(X) \subseteq K_\gamma(X)$;

iv) $\text{c-dim}(x) \leq \text{g-dim}(x)$, *for any* $x \in \widetilde{K}(X)$;

v) $K_+(X) \subseteq K_c(X)$.

This proposition shows that the γ-cone and the c-cone are approximations of the positive cone, more precisely, that they constitute upper bounds of the latter.

It turns out that the γ-cone and the c-cone coincide for torsion-free spaces, i.e. those spaces having no torsion in their integral cohomology.

PROPOSITION 3.3. *Let X be a connected finite CW-complex. If X is torsion-free, then*

$$K_\gamma(X) = K_c(X).$$

Proof. The result follows immediately from Proposition 2.2 and injectivity of the Chern character for a torsion-free space. □

It is worth mentioning that there is no general comparison statement for the γ-cone and the c-cone, i.e. there are spaces with torsion for which the γ-cone is not contained in the c-cone, and spaces with torsion for which the c-cone is not contained in the γ-cone. Moreover, there exist spaces for which the γ-cone and the c-cone strictly contain the positive cone (the product $S^4 \times S^4$ is such an example as we will later see). We now illustrate the situation by three examples.

EXAMPLES.

i) Let $j\colon BSU(3) \longrightarrow BU(3)$ be the map induced by the inclusion of the special unitary group $SU(3)$ in $U(3)$. Then the composition map

$$BSU(3) \xrightarrow{\ j\ } BU(3) \xrightarrow{\ \gamma_3\ } BU$$

lifts to a map $f\colon BSU(3) \longrightarrow BSU$. Consider W the homotopy fibre of f. It enters in a pull-back diagram

$$\begin{tikzcd}
SU \arrow[r, equal] \arrow[d, hook] & SU \arrow[d, hook] \\
W \arrow[r] \arrow[d, "\pi"'] & PBSU \arrow[d] \\
BSU(3) \arrow[r, "f"] & BSU
\end{tikzcd}$$

where $SU \simeq \Omega BSU \hookrightarrow PBSU \to BSU$ is the path-loop fibration of BSU. The Leray-Serre spectral sequence in cohomology for this fibration is well-known and maps via f^* to the corresponding spectral sequence for the fibration π. By Lemma 2.5, one has

$$f^*(\widetilde{c}_3) = j^* \circ \widetilde{\gamma}_3^*(c_3(\widetilde{\rho}_3)) = c_3(\widetilde{\gamma}_3) = 2\widetilde{c}_3 \,.$$

Similarly, one has $f^*(\widetilde{c}_2) = c_2(\widetilde{\gamma}_3)$, which is easily seen to vanish. For the cohomology of W in degree ≤ 6, letting $a_4 := \pi^*(\widetilde{c}_2)$ and $b_6 := \pi^*(\widetilde{c}_3)$, we have computed that $x_3^2 = 0$ and

$$H^{\leq 6}(W;\, \mathbf{Z}) = \mathbf{Z} \cdot 1 \oplus \mathbf{Z} \cdot x_3 \oplus \mathbf{Z} \cdot a_4 \oplus \mathbf{Z} \cdot x_5 \oplus \underbrace{\mathbf{Z} \cdot b_6}_{\cong \mathbf{Z}/2} \cong \mathbf{Z}^4 \oplus \mathbf{Z}/2 \,,$$

where $\deg(x_{2j+1}) = 2j+1$. The inclusion $i\colon Y := W^{[7]} \hookrightarrow W$ of the 7-skeleton of W induces an isomorphism in cohomology up to degree 6. If we let $x := i^* \circ \pi^* \circ j^*(\widetilde{\rho}_3) \in \widetilde{K}(Y)$, we find $c_3(x) = b_6 \neq 0$, whereas $\gamma^k(x) = 0$, for all $k \geq 3$. Indeed, this is clear for $k \geq 4$ since then $\gamma^k(\widetilde{\rho}_3) = 0$, and $\gamma^3(x) = 0$ because its classifying map is the composition $f \circ \pi \circ i$, which is homotopically trivial. Thus c-dim$(x) = 3$ and γ-dim$(x) \leq 2$. Consequently, Y is a connected finite CW-complex with a strict inclusion

$$K_c(Y) \subsetneq K_\gamma(Y) \,.$$

ii) Consider the Moore space $M = M(\mathbf{Z}/2,\, 5)$, i.e. the mapping cone of a continuous map $f\colon S^5 \xrightarrow{2} S^5$ of degree two, or more explicitly, $M = C_f = S^5 \cup_2 e^6$. The exact sequences in cohomology and in K-theory of the cofibration $S^5 \hookrightarrow M \overset{q}{\twoheadrightarrow} M/S^5 \simeq S^6$ give epimorphisms

$$q^* : \mathbf{Z} \cong H^6(S^6;\, \mathbf{Z}) \twoheadrightarrow H^6(M;\, \mathbf{Z}) \cong \mathbf{Z}/2$$

$$q^* : \mathbf{Z} \cong \widetilde{K}(S^6) \twoheadrightarrow \widetilde{K}(M) \cong \mathbf{Z}/2 \,.$$

Let x and a be suitable generators of $\widetilde{K}(S^6)$ and of $H^6(S^6; \mathbf{Z})$ respectively, and define $\bar{x} := q^*(x)$ and $\bar{a} := q^*(a)$. For obvious dimensional reasons, the Chern classes $c_1(\bar{x})$ and $c_2(\bar{x})$ vanish. Moreover, one has $c_3(\bar{x}) = q^*(c_3(x)) = q^*(2a) = 0$ (see Proposition 2.4), hence c-dim$(\bar{x}) = 0$. On the other hand, we have $\gamma^1(\bar{x}) = \bar{x} \neq 0$, so γ-dim$(\bar{x}) \geq 1$; more precisely, $\gamma^2(\bar{x})$ is $q^*(-S(3,2) \cdot x) = q^*(-3x) = \bar{x} \neq 0$ and $\gamma^3(\bar{x}) = q^*(2S(3,3) \cdot x) = 0$, so γ-dim$(\bar{x}) = 2$. Consequently, M is a connected finite CW-complex with a strict inclusion

$$K_\gamma(M) \subsetneq K_c(M)\,.$$

iii) Let $Z = Y \vee M$ be the wedge of the preceding two examples. It is a 7-dimensional finite connected CW-complex for which none of $K_\gamma(Z)$ and $K_c(Z)$ contains the other one. (The product $Y \times M$ would also do.)

To end the present section, we prove that the cones are semigroups and homotopy invariants.

PROPOSITION 3.5. *The positive cone, the γ-cone and the c-cone of a connected finite CW-complex X are sub-semigroups of $K(X)$ and homotopy invariants of X. Moreover, the positive cone is a sub-λ-semiring of $K(X)$.*

Proof. The homotopy invariance is obvious for the three cones. We have already mentioned in the preliminaries that the positive cone is a sub-semiring of $K(X)$. It is also clear that it is a sub-λ-semiring. The "exponentiality" of γ_t and of c (the total Chern class) immediately show that the γ-cone and the c-cone are sub-semigroups of $K(X)$. □

We do not know if in general the γ-cone and the c-cone are sub-λ-semirings of $K(X)$.

4. THE POSITIVE CONE OF THE SPHERES

We now intend to compute the positive cone of the spheres. For odd-dimensional spheres, there is nothing to do since $\widetilde{K}(S^{2n+1}) = 0$. Whereas for even-dimensional spheres, one has $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x \cong \mathbf{Z}$, so we only have to compute g-dim(lx) for all integers l.

By Proposition 2.4, we have

$$c(lx) = c(x)^l = (1 + (-1)^{n-1}(n-1)! \cdot a)^l = 1 + (-1)^{n-1} l(n-1)! \cdot a\,,$$

where a is the orientation class of S^{2n}. Therefore, by Proposition 3.2, we deduce that, for $l \neq 0$,

$$n = \text{c-dim}(lx) \leq \text{g-dim}(lx) \leq \dim(S^{2n})/2 = n\,,$$

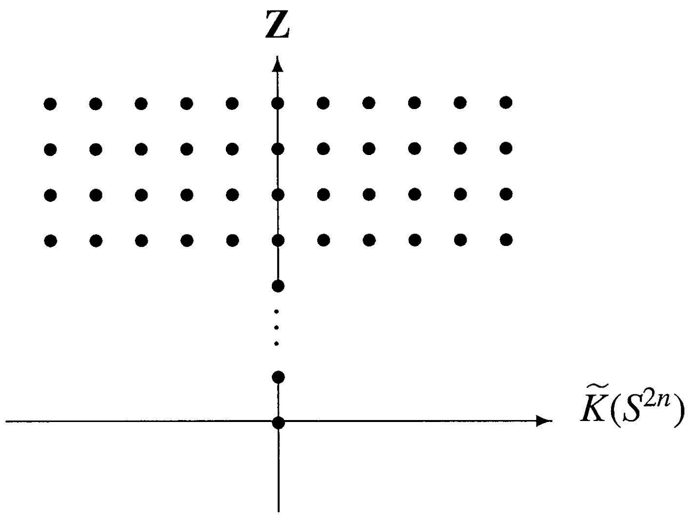
and this shows that $\text{c-dim}(lx) = \text{g-dim}(lx) = n$. The sphere S^{2n} being a torsion-free space, the following theorem follows from Proposition 3.3.

THEOREM 4.1. *Let x be a generator of $\widetilde{K}(S^{2n}) \cong \mathbf{Z}$. Then, for $l \in \mathbf{Z}$,*

$$\text{g-dim}(lx) = \begin{cases} 0 & \text{if } l = 0 \\ n & \text{otherwise.} \end{cases}$$

Moreover, the positive cone, the c-cone and the γ-cone of S^{2n} coincide:

$$K_+(S^{2n}) = K_c(S^{2n}) = K_\gamma(S^{2n}) = \mathbf{N} \times 0 \cup \{(l, x) \mid l \geq n\} \subset \mathbf{Z} \times \widetilde{K}(S^{2n})\,.$$



There is another, purely homotopic, proof of the theorem. It is based on Bott's celebrated results on the homotopy groups of $BU(n)$ and Serre's computation of the rational homotopy groups of spheres. Let us also present this proof. We have

$$[S^{2n}, BU(k)] = \pi_{2n}(BU(k)) \quad \text{and} \quad \widetilde{K}(S^{2n}) = [S^{2n}, BU] = \pi_{2n}(BU)\,.$$

Consider the long exact sequence of the fibration $BU(k) \xrightarrow{i_k} BU$:

$$\ldots \to \pi_{2n}(U/U(k)) \to \pi_{2n}(BU(k)) \xrightarrow{(i_k)_*} \pi_{2n}(BU) \to \pi_{2n-1}(U/U(k)) \to \ldots$$

The fibre $U/U(k)$ of i_k is $2k$-connected (see for example [MiTo], p. 216) and it follows that $(i_k)_*$ is an isomorphism for $n \leq k$. According to Bott [Bott2], we have $\pi_{2n}(BU) \cong \mathbf{Z}$. It is well-known that for $k < n$, the group $\pi_{2n}(BU(k))$ is finite. Let us however give a short proof of this result.

LEMMA 4.2. *For $m \geq 2k+1$, the group $\pi_m(BU(k))$ is finite.*

Proof. We fix $m \geq 3$. The fibration $BU(k-1) \longrightarrow BU(k)$, with fibre S^{2k-1}, yields the following long exact sequence in homotopy:

$$\ldots \to \pi_m(S^{2k-1}) \to \pi_m(BU(k-1)) \to \pi_m(BU(k)) \to \pi_{m-1}(S^{2k-1}) \to \ldots$$

By Serre [Serre], $\pi_j(S^{2k-1})$ is finite for $j \neq 2k-1$, and we can conclude by induction over k (with $k \geq 1$ and $2k+1 \leq m$), since when $k=1$, one has $\pi_m(BU(1)) = \pi_{m-1}(U(1)) = 0$ for $m \geq 3$. $\square$

From this, we now infer that the image of $(i_k)_*$ is zero for $k < n$. This implies that g-dim$(lx) = n$ when $l \neq 0$, and concludes the second proof.

REMARK 4.3.

i) Since we were motivated by Elliott's classification of unital C^*-algebras of type AF by means of their K-theory, their positive cone and the K-theory class $[1]$ of the unit (see [Bla1]), it is important to single out the fact that the positive cone of S^{2n} and that of S^{2m} are non-isomorphic as monoids if n is different from m. (There is no need here to distinguish the K-theory class 1 of the trivial one-dimensional bundle.) Let us provide a short proof of this claim. For $n \geq 1$, let M_n denote the positive cone of S^{2n} (identified as above with a sub-monoid of $\mathbf{Z}^2$, in order to designate its elements). The abelian monoid M_n has a minimal set A_n of generators, in other words a generating set (as a monoid) that is contained in any other generating set, namely

$$A_n = \{(0,1)\} \cup \{(k,n) \mid k \in \mathbf{Z} \setminus \{0\}\}.$$

Now, consider the function $\sigma \colon A_n \longrightarrow \{2, 3, \ldots\}$ defined, for $x \in A_n$, by

$$\sigma(x) := \min \{l \geq 2 \mid lx \text{ decomposes as a sum of elements of } A_n \setminus \{x\}\}.$$

Clearly, such an l exists for any $x \in A_n$ and $\sigma(A_n) = \{2, 2n\}$. Since A_n and σ are isomorphism invariants of M_n, this proves our claim.

ii) For odd-dimensional spheres the positive cone is "trivial"; in other words, $K(S^{2n-1}) = \mathbf{Z}$ and $K_+(S^{2n-1}) = \mathbf{N}$.

5. FURTHER PROPERTIES OF THE CONES

We now investigate naturality properties and behaviour under products of the positive cone, the γ-cone and the c-cone.

The following result is obvious.

PROPOSITION 5.1. *Let $f\colon X \longrightarrow Y$ be a map between connected finite CW-complexes. Let $f^*\colon K(Y) \longrightarrow K(X)$ be the λ-homomorphism induced by f. Then, for any $y \in \widetilde{K}(Y)$, one has*

$$\text{g-dim}(f^*(y)) \leq \text{g-dim}(y)$$

$$\gamma\text{-dim}(f^*(y)) \leq \gamma\text{-dim}(y)$$

$$\text{c-dim}(f^*(y)) \leq \text{c-dim}(y)\,,$$

and in particular,

$$f^*(K_+(Y)) \subseteq K_+(X)$$

$$f^*(K_\gamma(Y)) \subseteq K_\gamma(X)$$

$$f^*(K_c(Y)) \subseteq K_c(X)\,.$$

Furthermore, if f^* is an isomorphism, then

$$f^*(K_\gamma(Y)) = K_\gamma(X)\,.$$

For the next corollary we need a new definition.

DEFINITION 5.2. Let X and Y be two connected finite CW-complexes. A map $f\colon X \longrightarrow Y$ is called a *K^0-equivalence* (or *K-equivalence* for short) if there exists a map $g\colon Y \longrightarrow X$ such that, on the level of the K^0-groups,

$$f^* \circ g^* = Id_{K^0(X)} \qquad \text{and} \qquad g^* \circ f^* = Id_{K^0(Y)}\,.$$

Note that a K-equivalence is *not* necessarily a homotopy equivalence: there are homotopically non-trivial (i.e. non-contractible) finite CW-complexes X for which $\widetilde{K}(X) = 0 = \widetilde{K}(pt)$; see example iii) below.

PROPOSITION 5.3. *If $f\colon X \longrightarrow Y$ is a K-equivalence, then f induces the following isomorphisms of semigroups:*

$$K_+(Y) \overset{f^*}{\cong} K_+(X) \quad \textit{and} \quad K_\gamma(Y) \overset{f^*}{\cong} K_\gamma(X)\,.$$

Proof. Applying Proposition 5.1 twice, we get (in the notations of Definition 5.2)

$$K_+(X) = f^* \circ g^*(K_+(X)) \subseteq f^*(K_+(Y)) \subseteq K_+(X)\,.$$

This establishes the first isomorphism, whereas the second one is obvious. □

The following result is more technical to state.

COROLLARY 5.4. *Let X and Y be two connected finite CW-complexes. Assume that $K^1(X) = 0$ and that $\widetilde{K}^0(Y) = 0$. Then the projection $p \colon X \times Y \longrightarrow X$ induces isomorphisms*

$$K_+(X) \overset{p^*}{\cong} K_+(X \times Y) \quad \textit{and} \quad K_\gamma(X) \overset{p^*}{\cong} K_\gamma(X \times Y)\,.$$

Proof. Invoking the Künneth theorem for K-theory, our hypotheses imply that $p^* \colon K^0(X) \longrightarrow K^0(X \times Y)$ is an isomorphism with inverse i^*, where i is the inclusion of X in $X \times Y$. Consequently, p^* is a K-equivalence. $\square$

The following is a useful result.

PROPOSITION 5.5. *Let X and Y be connected finite CW-complexes. Assume that the positive cone and the γ-cone of Y coincide, and let $f \colon X \longrightarrow Y$ be a map inducing an isomorphism $f^* \colon K(Y) \longrightarrow K(X)$. Then f induces an isomorphism of positive cones, and the γ-cone of X coincides with the positive cone:*

$$K_+(Y) \overset{f^*}{\cong} K_+(X) = K_\gamma(X)\,.$$

Proof. By Proposition 5.1 we have $f^*(K_+(Y)) = f^*(K_\gamma(Y)) = K_\gamma(X)$ and $f^*(K_+(Y)) \subseteq K_+(X)$, hence $K_\gamma(X) \subseteq K_+(X)$. We conclude with iii) of Proposition 3.2. $\square$

EXAMPLES.

i) Let X be a connected finite CW-complex of dimension ≤ 3. Since for suitable CW-decompositions, one has $BU(1)^{[3]} = BU^{[3]}$ and since $BU(1) = \mathbf{C}P^\infty = K(\mathbf{Z}, 2)$, any $x \in \widetilde{K}(X) = [X, BU]$ lifts to a class in $[X, BU(1)]$, giving an isomorphism $\widetilde{K}(X) \cong H^2(X; \mathbf{Z})$ mapping x to $c_1(x)$. It follows that the positive cone coincides with the c-cone and is given by

$$K_+(X) = \mathbf{N} \times \{0\} \ \cup \ \mathbf{N}^* \times \widetilde{K}(X) \subset \mathbf{Z} \times \widetilde{K}(X)\,.$$

ii) Example i) applies to a closed oriented surface Σ_g of genus g. Since it is torsion-free, its positive cone coincides with its c-cone and with its γ-cone. Moreover, let $f \colon \Sigma_g \longrightarrow S^2$ be a map of degree 1 (it exists, since both the 2-sphere and Σ_g are quotients of the square $[0, 1]^2$). Then f not only induces an isomorphism in K-theory, but also an isomorphism of positive cones, as follows from Proposition 5.1.

iii) Let X and Y be the Moore spaces $M(\mathbf{Z}/3, 2q+11) = S^{2q+11} \cup_3 e^{2q+12}$ and $M(\mathbf{Z}/3, 2q-1) = S^{2q-1} \cup_3 e^{2q}$ respectively. In [Adams], Adams shows that for q large enough, there exists a map $A\colon X = \Sigma^{12} Y \longrightarrow Y$ such that the induced map $A^*\colon \widetilde{K}(Y) \longrightarrow \widetilde{K}(X)$ is an isomorphism (take $p = m = 3$, $f = 1$ and $r = 6$ in Theorem 1.7 and in Lemmas 12.4 and 12.5 of [Adams]). Therefore, A is a K-isomorphism between simply connected finite CW-complexes, but it is *not* a homotopy equivalence. The mapping cone C_A is a non-contractible finite CW-complex with $\widetilde{K}(C_A) = 0$. (It is non-contractible because its homology is non-trivial.)

iv) In [GrMo], pp. 203-206, a CW-complex $X = (S^1 \vee S^2) \cup e^3$ is defined, with the property that the inclusion $i\colon S^1 = X^{[1]} \hookrightarrow X$ of the 1-skeleton induces an isomorphism in integral homology (and on the level on fundamental groups); however, i is *not* a homotopy equivalence since $\pi_2(X) \neq 0$. Consequently, by the universal coefficient theorem (see Corollary V.7.2 in [Bred]), i induces an isomorphism in integral cohomology, and, by a direct application of the Atiyah-Hirzebruch spectral sequence, also in K-theory. In particular, i is a K-equivalence, but *not* an equivalence. (As C_A in the preceding example, the quotient space $X/X^{[1]}$ has vanishing $\widetilde{K}$, however it is the closed 3-ball and is therefore contractible.)

Let us finally mention that in [Matt], the positive cone, the c-cone and the γ-cone are also studied from the rational point of view, and rational K-theory is considered.

6. The cones of the products $S^n \times S^{2m-1}$

In this section, we will compute the cones for the products $S^{2n} \times S^{2m-1}$ and $S^{2n-1} \times S^{2m-1}$.

We begin with $S^{2n} \times S^{2m-1}$. Since $\widetilde{K}(S^{2m-1}) = 0$ and $K^1(S^{2n}) = 0$, the answer immediately follows from Proposition 5.5.

THEOREM 6.1. *The projection $p\colon S^{2n} \times S^{2m-1} \longrightarrow S^{2n}$ induces an isomorphism of positive cones, and, for $S^{2n} \times S^{2m-1}$, the γ-cone and the c-cone coincide with the positive cone:*

$$K_+(S^{2n}) \overset{p^*}{\cong} K_+(S^{2n} \times S^{2m-1}) = K_\gamma(S^{2n} \times S^{2m-1}).$$

We now turn to the product $S^{2n-1} \times S^{2m-1}$. From the six-term exact sequence of the pair $(S^{2n-1} \times S^{2m-1}, S^{2n-1} \vee S^{2m-1})$, with quotient the smash product $S^{2n-1} \wedge S^{2m-1}$ homeomorphic to $S^{2m+2n-2}$, we get an isomorphism

$$q^* \colon \widetilde{K}(S^{2m+2n-2}) \longrightarrow \widetilde{K}(S^{2n-1} \times S^{2m-1})$$

induced by the quotient map $q \colon S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$. By Theorem 4.1, the space $Y = S^{2n+2m-2}$ satisfies the hypothesis of Proposition 5.5 and we deduce the

THEOREM 6.2. *The map $q \colon S^{2n-1} \times S^{2m-1} \longrightarrow S^{2m+2n-2}$ induces an isomorphism of positive cones, and, for $S^{2n-1} \times S^{2m-1}$, the γ-cone and the c-cone coincide with the positive cone:*

$$K_+(S^{2m+2n-2}) \overset{q^*}{\cong} K_+(S^{2n-1} \times S^{2m-1}) = K_\gamma(S^{2n-1} \times S^{2m-1}).$$

REMARK 6.3. According to Blackadar ([Bla2], 6.10.2), the positive cone of the n-torus $(S^1)^n$ has been partially computed by Villadsen.

7. THE γ-CONE OF $S^{2n} \times S^{2m}$ AND THE POSITIVE CONE OF $S^2 \times S^{2n}$

The positive cone was rather easy to compute for a product of an odd-dimensional sphere by any sphere, whereas the case of a product of two even-dimensional spheres is much more involved. On the other hand, the γ-cone of such a product is in the scope of the present notes. We perform this calculation by computing the c-cone and appealing to Proposition 3.3.

By the Künneth theorem, we have an isomorphism

$$K(S^{2n}) \otimes K(S^{2m}) \longrightarrow K(S^{2n} \times S^{2m}), \ \xi \otimes \eta \longmapsto p^*(\xi) \cdot q^*(\eta),$$

where p and q are the projections onto the factors. Writing $\widetilde{K}(S^{2n}) = \mathbf{Z} \cdot x_1$ and $\widetilde{K}(S^{2m}) = \mathbf{Z} \cdot x_2$, and letting $y_1 := p^*(x_1)$ and $y_2 := q^*(x_2)$, we deduce that

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbf{Z} \cdot y_1 \oplus \mathbf{Z} \cdot y_2 \oplus \mathbf{Z} \cdot y_1 y_2 .$$

The product structure on $\widetilde{K}(S^{2n} \times S^{2m})$ is given by $y_1^2 = 0$ and $y_2^2 = 0$. One has $y_1 y_2 = \pi^*(y)$, where $\pi \colon S^{2n} \times S^{2m} \longrightarrow S^{2n} \wedge S^{2m} \cong S^{2n+2m}$ and y is a suitable generator of $\widetilde{K}(S^{2n+2m})$. Let $i \colon S^{2n} \hookrightarrow S^{2n} \times S^{2m}$ and $j \colon S^{2m} \hookrightarrow S^{2n} \times S^{2m}$ be the inclusions. One has $i^*(y_1) = x_1$ and $j^*(y_2) = x_2$, and (by Theorem 4.1 and a double application of Proposition 5.1), for any $k \in \mathbf{Z} \setminus \{0\}$, one has g-dim$(ky_1)$ = g-dim$(kx_1) = n$; similarly g-dim(ky_2) = g-dim$(kx_2) = m$. This justifies that, from now on, we write x_1 and x_2 for y_1 and y_2 respectively.

Let $a_1 \in H^{2n}(S^{2n}; \mathbf{Z})$ and $a_2 \in H^{2m}(S^{2m}; \mathbf{Z})$ be suitable generators (referring to Proposition 2.4). As before, it is justified to write

$$\widetilde{H}^*(S^{2n} \times S^{2m}; \mathbf{Z}) = \mathbf{Z} \cdot a_1 \oplus \mathbf{Z} \cdot a_2 \oplus \mathbf{Z} \cdot a_1 a_2 .$$

Let us assume $n \leq m$. Consider an element $x = ax_1 + bx_2 + lx_1x_2$ in the group $\widetilde{K}(S^{2n} \times S^{2m})$. For the Chern classes, invoking Proposition 2.4 and "exponentiality" of the total Chern class, we compute

$$\begin{aligned} c(x) &= c(ax_1)c(bx_2)c(lx_1x_2) \\ &= 1 + (-1)^{n-1}a(n-1)! \cdot a_1 + (-1)^{m-1}b(m-1)! \cdot a_2 \\ &\qquad + (-1)^{n+m}(ab(n-1)!\,(m-1)! - l(n+m-1)!) \cdot a_1 a_2 . \end{aligned}$$

This immediately gives the γ-cone (which coincides with the c-cone) in terms of the γ-dimension function.

THEOREM 7.1. *For $n \leq m$, the γ-dimension on $\widetilde{K}(S^{2n} \times S^{2m})$ is given as follows: for $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$, one has*

$$\gamma\text{-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ n & \text{if } a \neq 0,\ b = l = 0 \\ m & \text{if } b \neq 0,\ l = ab\,(n-1)!\,(m-1)!/(n+m-1)! \\ n+m & \text{if } l \neq ab\,(n-1)!\,(m-1)!/(n+m-1)! \end{cases}$$

Moreover, for $k \neq 0$, one has

$$\text{g-dim}(kx_1) = n \quad \text{and} \quad \text{g-dim}(kx_2) = m .$$

This theorem allows us to give some interesting information on the positive cone of the product $S^{2n} \times S^{2m}$. We will state the result as Theorem 8.2 in the following section, because the tools developed there allow one to make a crucial improvement.

Combined with Theorem 2.3, Theorem 7.1 enables one to compute completely the positive cone of $S^2 \times S^{2n}$.

THEOREM 7.2. *For the product $S^2 \times S^{2n}$, we have*

$$K_+(S^2 \times S^{2n}) = K_c(S^2 \times S^{2n}) = K_\gamma(S^2 \times S^{2n}) .$$

The latter is given by Theorem 7.1.

8. THE WHITEHEAD PRODUCT AND THE POSITIVE CONE

We will establish an interesting connection between the positive cone of a product $S^{2n} \times S^{2m}$ and the Whitehead product structure on the homotopy groups of the spaces $BU(k)$. As an application we will get some precise information on the positive cone of $S^{2n} \times S^{2m}$.

Let us first recall the basic properties of the Whitehead product (the reader may refer to [White]). The product $S^p \times S^q$ has a cell structure obtained by attaching a $(p+q)$-cell to $S^p \vee S^q$. More precisely, there exists a suitable pointed map $f_0 \colon S^{p+q-1} \longrightarrow S^p \vee S^q$ such that $S^p \times S^q$ is homeomorphic to the mapping cone of f_0:

$$S^p \times S^q \cong C_{f_0} = (S^p \vee S^q) \cup_{f_0} e^{p+q}$$

Given a pointed map $g = \alpha \vee \beta \colon S^p \vee S^q \longrightarrow X$, where X is a CW-complex, there exists (up to homotopy) an extension $\bar{g} \colon S^p \times S^q \longrightarrow X$ of g if and only if the composition $g \circ f_0$ is homotopically trivial. Now, considering α and β as elements of the homotopy groups $\pi_p(X)$ and $\pi_q(X)$ respectively, the composition $(\alpha \vee \beta) \circ f_0$ determines an element in the homotopy group $\pi_{p+q-1}(X)$. This defines a map

$$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)\,, \ (\alpha,\, \beta) \longmapsto [\alpha,\, \beta] := (\alpha \vee \beta) \circ f_0\,,$$

which by definition is the Whitehead product. One can show that it is $\mathbf{Z}$-bilinear (provided that $p,\, q \geq 2$), i.e.

$$[\alpha_1 + \alpha_2,\, \beta] = [\alpha_1,\, \beta] + [\alpha_2,\, \beta] \quad \text{and} \quad [\alpha,\, \beta_1 + \beta_2] = [\alpha,\, \beta_1] + [\alpha,\, \beta_2]\,.$$

Moreover, the Whitehead product is natural with respect to pointed maps, i.e. if $f \colon X \longrightarrow Y$ is a pointed map between CW-complexes, then

$$[f_*(\alpha), f_*(\beta)] = f_*([\alpha,\, \beta])\,.$$

We now want to study the case where $X = BU(l)$. Let x_1 and x_2 be two generators of $\widetilde{K}(S^{2n})$ and $\widetilde{K}(S^{2m})$ respectively, and assume $1 \leq n \leq m$. By Theorem 4.1, we know that g-dim$(x_1) = n$ and that g-dim$(x_2) = m$. Letting $q \geq m$, we consider x_1 and x_2 as maps from S^{2n} (respectively S^{2m}) to BU that lift to $BU(q)$. The element $x_1 + x_2$ of $\widetilde{K}(S^{2n} \vee S^{2m}) = \widetilde{K}(S^{2n}) \oplus \widetilde{K}(S^{2m})$ can be represented by the map $x_1 \vee x_2 \colon S^{2n} \vee S^{2m} \longrightarrow BU$, and it also lifts to a map $z \colon S^{2n} \vee S^{2m} \longrightarrow BU(q)$.

CLAIM. *For $k \in \{m, m+1, \ldots, m+n-1\}$, there is no extension of $z = x_1 \vee x_2 \colon S^{2n} \vee S^{2m} \longrightarrow BU(k)$ to a map $S^{2n} \times S^{2m} \longrightarrow BU(k)$.*

Let $y \colon S^{2n} \times S^{2m} \longrightarrow BU(s)$ be an extension of z for some $s \geq m$. Let x be the composition of y with the map $i_s \colon BU(s) \longrightarrow BU$. This means that g-dim$(x) \leq s$ and that $\iota^*(x) = x_1 + x_2 \in \widetilde{K}(S^{2n} \vee S^{2m})$, where ι is the inclusion of $S^{2n} \vee S^{2m}$ in the product $S^{2n} \times S^{2m}$. Recall that $(\iota^*)^{-1}(x_1 + x_2) = x_1 + x_2 + \mathbf{Z} \cdot x_1 x_2 \subset \widetilde{K}(S^{2n} \times S^{2m})$. So, there exists an integer l such that $x = x_1 + x_2 + l x_1 x_2$, and consequently

$$\gamma^{n+m}(x) = (-1)^{n+m-1}(l\,(n+m-1)! - (n-1)!\,(m-1)!) \cdot x_1 x_2 \neq 0\,.$$

We see that $s \geq$ g-dim$(x) \geq \gamma$-dim$(x) \geq n+m$. This proves the claim.

As a direct consequence, by considering x_1 and x_2 as elements (in fact generators) of $\pi_{2n}(BU(k))$ and $\pi_{2m}(BU(k))$ respectively, we get the following result on the Whitehead product:

$$[x_1, x_2] \neq 0 \text{ in } \pi_{2n+2m-1}(BU(k))\,, \quad \text{for } m \leq k < n+m\,.$$

We would now like to get some information on the order of $[x_1, x_2]$ in the homotopy group $\pi_{2n+2m-1}(BU(k))$. By $\mathbf{Z}$-bilinearity of the Whitehead product, we have $ab[x_1, x_2] = [ax_1, bx_2]$ for any integers a and b. Replacing x_1 by ax_1 and x_2 by bx_2 in the preceding computation (in particular $x = ax_1 + bx_2 + lx_1x_2$ for some l), one easily verifies that

$$(*) \qquad \left.\begin{array}{l} ab[x_1, x_2] = 0 \\ \text{in } \pi_{2n+2m-1}(BU(k)) \\ \text{for } m \leq k < n+m \end{array}\right\} \Longrightarrow l\,(n+m-1)! - ab(n-1)!\,(m-1)! = 0$$

and this implies that ab is a multiple of $(n+m-1)!/((n-1)!\,(m-1)!)$. Notice that $[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$ has to be a torsion element. Indeed, by Lemma 4.2, the group $\pi_{2n+2m-1}(BU(m))$ is finite, and the result follows from naturality of the Whitehead product. (In fact, one can show that any group $\pi_{2i+1}(BU(j))$ is finite; this is proved like Lemma 4.2, by appealing to a result of Borel and Hirzebruch: see Remark i) in Section 9.) We have thus obtained the following theorem.

THEOREM 8.1. *Let $1 \leq n \leq m$ and $m \leq k < n+m$. Let x_1 and x_2 be generators of the homotopy groups $\pi_{2n}(BU(k)) \cong \mathbf{Z}$ and $\pi_{2m}(BU(k)) \cong \mathbf{Z}$ respectively. Then the Whitehead product $[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$ is non-zero. Moreover, its order is a multiple of* $\dfrac{(n+m-1)!}{(n-1)!\,(m-1)!}$.

By the implication $(*)$, if $ab[x_1, x_2] = 0$ in $\pi_{2n+2m-1}(BU(k))$ for some k with $m \leq k < n+m$, then for $l = ab\,(n-1)!\,(m-1)!/(n+m-1)!$, the geometric dimension of $x := ax_1 + bx_2 + lx_1x_2$ is $\leq k$ (and for any other value of l, g-dim(x) is $m+n$, provided that $ab \neq 0$). Surprisingly, this condition only depends on l and on the product ab. Consequently, from Theorem 2.3 together with Theorem 7.1, we obtain the following result.

THEOREM 8.2. *The geometric dimension on $\widetilde{K}(S^{2n} \times S^{2m})$, with $n \leq m$, is given as follows: for $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$,*

$$\text{g-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ n & \text{if } a \neq 0,\ b = l = 0 \\ m & \text{if } a = 0,\ b \neq 0,\ l = 0 \\ s(ab) & \text{if } b \neq 0,\ l = ab\,(n-1)!\,(m-1)!/(n+m-1)! \\ n+m & \text{if } l \neq ab\,(n-1)!\,(m-1)!/(n+m-1)! \end{cases}$$

where $s(ab) \in \{m, m+1, \ldots, n+m-1\}$ only depends on the product ab (for fixed n and m).

As a direct consequence of Theorems 8.1 and 8.2, we have

COROLLARY 8.3. *The order of the Whitehead product $[x_1, x_2]$ in $\pi_{2n+2m-1}(BU(n+m-1))$ is exactly $(n+m-1)!/((n-1)!\,(m-1)!)$.*

REMARK 8.4.

i) This result has been established only using information on the γ-cone of $S^{2n} \times S^{2m}$ (and Serre's theorem on the rational homotopy of spheres). If one is able to compute its positive cone, one then can easily compute the exact order of $[x_1, x_2]$ in the various homotopy groups $\pi_{2n+2m-1}(BU(k))$, for $m \leq k < n+m$: it is given by

$$\min\left\{ l \geq 1 \;\middle|\; \text{g-dim}\left(l \frac{(n+m-1)!}{(n-1)!\,(m-1)!} x_1 + x_2 + lx_1x_2 \right) \leq k \right\}.$$

ii) In 1960, Bott [Bott3] has proved Corollary 8.3 by different methods.

9. The positive cone of some products of even-dimensional spheres

In this section, using known results from the theory of homotopy groups of spheres, we compute the positive cone of $S^4 \times S^4$, $S^4 \times S^6$, $S^6 \times S^6$ and $S^6 \times S^8$. This computation will in particular show that the positive cone and the γ-cone do not coincide for $S^4 \times S^4$. Keeping notations as in Section 7, we describe the positive cone in terms of the geometric dimension function.

A) We start with the case of $S^4 \times S^4$.

THEOREM 9.1. *The geometric dimension on $\widetilde{K}(S^4 \times S^4)$ is given as follows: for $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^4 \times S^4)$, one has*

$$\text{g-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ 2 & \text{if } a \neq 0,\ b = l = 0 \\ 2 & \text{if } b \neq 0,\ l = ab/6,\ l \text{ even} \\ 3 & \text{if } b \neq 0,\ l = ab/6,\ l \text{ odd} \\ 4 & \text{if } l \neq ab/6 \end{cases}$$

Proof. Theorem 8.2 reduces the problem to the computation of the function $s = s(ab)$, i.e. to calculating g-dim(x) for the particular stable classes $x = ax_1 + bx_2 + (ab/6)x_1x_2$ (where ab is a multiple of 6), or equivalently the order of $[x_1, x_2]$ in both groups $\pi_7(BU(3))$ and $\pi_7(BU(2))$ (with a little abuse of notation, we denote both Whitehead products by the same symbol). By Samelson [Sam], one has

$$\pi_7(BU(2)) \cong \pi_6(U(2)) \cong \pi_6(SU(2)) \cong \pi_6(S^3) \cong \mathbf{Z}/12,$$

precisely generated by $[x_1, x_2]$. This shows that for these particular values of x, g-dim$(x) = 2$ if and only if ab is a multiple of 12. This completes the proof. □

REMARK 9.2.

i) Borel and Hirzebruch in [BoHi] (p. 355), applying Bott's results of [Bott1], have proved that

$$\pi_{2n+1}(BU(n)) \cong \pi_{2n}(SU(n)) \cong \mathbf{Z}/n! \quad (n \geq 2),$$

hence $\pi_7(BU(3)) \cong \mathbf{Z}/6$. Moreover, Corollary 8.3 shows that the order of $[x_1, x_2]$ in $\pi_7(BU(3))$ is 6; it is consequently a generator.

ii) As already alluded to, we have just proved that $S^4 \times S^4$ has its positive cone strictly contained in its γ-cone, although it is a torsion-free space.

B) As for $S^4 \times S^4$, classical results from the theory of homotopy groups of the unitary groups allow one to compute the positive cone of $S^4 \times S^6$. In this case, it coincides with the γ-cone.

THEOREM 9.3. *For the product $S^4 \times S^6$, one has*

$$K_+(S^4 \times S^6) = K_c(S^4 \times S^6) = K_\gamma(S^4 \times S^6).$$

The latter is described in Theorem 7.1.

Proof. By Lundell's tables [Lun] (see also [Mim]) and by Remark i) above, one has

$$\pi_9(BU(3)) \cong \mathbf{Z}/12 \quad \text{and} \quad \pi_9(BU(4)) \cong \mathbf{Z}/24.$$

Corollary 8.3 shows that $[x_1, x_2]$ is of order 12 in $\pi_9(BU(4))$. By naturality of the Whitehead product, the homomorphism $j_* = \pi_9(j)$, induced by the map $j\colon BU(3) \longrightarrow BU(4)$, takes the product $[x_1, x_2] \in \pi_9(BU(3))$ to $[x_1, x_2] \in \pi_9(BU(4))$. This implies that $[x_1, x_2]$ is of order 12 in $\pi_9(BU(3))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_9(BU(3))$ precisely when it is zero in $\pi_9(BU(4))$. Together with Theorem 8.2, this completes the proof. □

REMARK 9.4. This proof shows in particular that $[x_1, x_2]$ is a generator of $\pi_9(BU(3)) \cong \mathbf{Z}/12$ and that the map $j_*\colon \pi_9(BU(3)) \longrightarrow \pi_9(BU(4))$ is injective.

C) By similar methods, we now show that the positive cone and the γ-cone coincide for $S^6 \times S^6$ and then for $S^6 \times S^8$.

THEOREM 9.5. *For the product $S^6 \times S^6$, one has*

$$K_+(S^6 \times S^6) = K_c(S^6 \times S^6) = K_\gamma(S^6 \times S^6).$$

The latter is given by Theorem 7.1.

Proof. By Lundell's tables [Lun] (see also [Mim]), one has

$$\pi_{11}(BU(3)) \cong \mathbf{Z}/30 \quad \text{and} \quad \pi_{11}(BU(5)) \cong \mathbf{Z}/120.$$

Corollary 8.3 shows that $[x_1, x_2]$ is of order 30 in $\pi_{11}(BU(5))$. By naturality, the map $j_* = \pi_{11}(j)$, induced by $j\colon BU(3) \longrightarrow BU(5)$, takes the Whitehead product $[x_1, x_2] \in \pi_{11}(BU(3))$ to $[x_1, x_2] \in \pi_{11}(BU(5))$. This implies that $[x_1, x_2]$ is of order 30 in $\pi_{11}(BU(3))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_{11}(BU(3))$ precisely when it is zero in $\pi_{11}(BU(5))$. Together with Theorem 8.2, this completes the proof. □

REMARK 9.6.

i) This shows that $[x_1, x_2]$ generates $\pi_{11}(BU(3)) \cong \mathbf{Z}/30$ and that the map $j_*\colon \pi_{11}(BU(3)) \longrightarrow \pi_{11}(BU(5))$ is injective.

ii) We were also able to prove this theorem without appealing to results on homotopy groups of $BU(n)$. Using spectral sequence arguments, we have computed the first few stages of the Moore-Postnikov tower of the map $BSU(3) \longrightarrow BSU(5)$. This computation, being extremely lengthy, is not given here (see [Matt]).

Now we move on to the product $S^6 \times S^8$.

THEOREM 9.7. *For the product $S^6 \times S^8$, one has*

$$K_+(S^6 \times S^8) = K_c(S^6 \times S^8) = K_\gamma(S^6 \times S^8).$$

The latter is described in Theorem 7.1.

Proof. By Lundell's tables [Lun] (see also [Mim]), one has

$$\pi_{13}(BU(4)) \cong \mathbf{Z}/60 \quad \text{and} \quad \pi_{13}(BU(6)) \cong \mathbf{Z}/720.$$

Corollary 8.3 shows that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(6))$. By naturality, the map $j_* = \pi_{13}(j)$, induced by $j\colon BU(4) \longrightarrow BU(6)$, takes the Whitehead product $[x_1, x_2] \in \pi_{13}(BU(4))$ to $[x_1, x_2] \in \pi_{13}(BU(6))$. This implies that $[x_1, x_2]$ is of order 60 in $\pi_{13}(BU(4))$ too, and that $[ax_1, bx_2]$ vanishes in $\pi_{13}(BU(4))$ precisely when it is zero in $\pi_{13}(BU(6))$. Together with Theorem 8.2, this completes the proof. $\square$

REMARK 9.8. The proof shows that $[x_1, x_2]$ is a generator of the group $\pi_{13}(BU(4)) \cong \mathbf{Z}/60$ and that the map $j_*\colon \pi_{13}(BU(4)) \longrightarrow \pi_{13}(BU(6))$ is injective.

10. "GAPS IN COHOMOLOGY" AND THE γ-CONE

In the present section, we are interested in spaces having a "gap in cohomology", more precisely we look at spaces obtained by attaching a single large-dimensional cell to a finite CW-complex Y. For such spaces, the integral cohomology is zero between the dimension of Y and the top-dimensional class. The products $S^n \times S^m$ are typical examples (see Section 8). For this kind of spaces, the c-cone obviously cannot give information in the dimensions corresponding to the gap. At first sight, one could think that the γ-cone is more powerful in this range. Unfortunately, this is not the case: we show that the γ-cone (or equivalently the γ-dimension function) is also "blind" in some sense. Here is the precise statement.

PROPOSITION 10.1. *Let Y be a connected finite CW-complex of dimension $\leq 2n$, and let $X = C_f = Y \cup_f e^{2n+2m}$ be the mapping cone of a map $f\colon S^{2n+2m-1} \longrightarrow Y$, with $m \geq 1$. Then, for $x \in \widetilde{K}(X)$, one has*

$$\gamma^{n+m}(x) = 0 \implies \gamma^{n+l}(x) = 0 \quad \textit{for all} \quad l = 1, \ldots, m\,.$$

In other words, if γ-dim$(x) < n+m$, then γ-dim$(x) \leq n$.

Proof. By assumption, one has $H^k(X;\, \mathbf{Z}) = 0$ for $2n < k < 2n+2m$ and $H^{2n+2m}(X;\, \mathbf{Z}) \cong \mathbf{Z}$. Let $x \in \widetilde{K}(X)$ such that $\gamma^{n+m}(x) = 0$. By Proposition 2.2, keeping the same notation, we have

$$ch(\gamma^k(x)) = \bar{c}_k(x) + P_{k+1}(\bar{c}_1(x), \, \ldots, \, \bar{c}_{n+m}(x))\,,$$

and $0 = ch(\gamma^{n+m}(x)) = \bar{c}_{n+m}(x)$. Due to the "gap" in the cohomology of X, we find that, for $k > n$, we have

$$ch(\gamma^k(x)) = 0\,.$$

By the particular cohomological properties of X, the Chern character is injective for elements of filtration $> n$ in $\widetilde{K}(X)$ (see [AtHi]). Being zero or of filtration $\geq k$ (as Proposition 2.2 shows), $\gamma^k(x)$ has to vanish for $k > n$. This concludes the proof. $\square$

11. A "DOUBLING FORMULA" FOR STIRLING NUMBERS OF THE SECOND KIND

In the present section, we calculate the γ-operations for the product $S^{2n} \times S^{2m}$. From this computation and Proposition 10.1, we deduce again the γ-cone, as appearing in Theorem 7.1. This example illustrates that computing the c-cone is in general easier than computing the γ-cone. On the other hand, the latter calculation leads to an interesting "doubling formula" for Stirling numbers of the second kind. We will also conjecture the analogous formula for Stirling numbers of the first kind.

Keeping notations as in Section 7, we have

$$\widetilde{K}(S^{2n} \times S^{2m}) = \mathbf{Z} \cdot x_1 \oplus \mathbf{Z} \cdot x_2 \oplus \mathbf{Z} \cdot x_1 x_2 .$$

We still assume $n \leq m$. Using the known γ-operations for even-dimensional spheres, one can easily calculate γ^k for $S^{2n} \times S^{2m}$: For $x = ax_1 + bx_2 + lx_1x_2$, one has clearly $\gamma^k(x) = \gamma^k(ax_1 + bx_2) + \gamma^k(lx_1x_2)$ and this allows one to compute

$$\gamma^{m+q}(x) = (-1)^{m+q-1}(m+q-1)! \cdot \left(l\, S(m+n, m+q) - ab \sum_{k=q}^{n} \frac{S(n, k)S(m, m+q-k)}{k\binom{m+q-1}{k}} \right) \cdot x_1 x_2$$

for $q \geq 1$; in particular

$$\gamma^{n+m}(x) = (-1)^{n+m-1}(l\,(n+m-1)! - ab\,(n-1)!\,(m-1)!) \cdot x_1 x_2 .$$

For γ^m, we have to distinguish the case $n = m$ from the case $n < m$. One gets

$$\gamma^m(x) = (-1)^{m-1} a\,(m-1)! \cdot x_1 \;+\; (-1)^{m-1} b\,(m-1)! \cdot x_2 \;+\; (-1)^{m-1} \cdot (m-1)! \left(l\, S(2m, m) - ab \sum_{k=1}^{m-1} \frac{S(m, k)S(m, m-k)}{k\binom{m-1}{k}} \right) \cdot x_1 x_2$$

when $n = m$, whereas

$$\gamma^m(x) = (-1)^{m-1} b\,(m-1)! \cdot x_2 \;+\; (-1)^{m-1}(m-1)! \cdot \left(l\, S(n+m, m) - ab \sum_{k=1}^{m-1} \frac{S(n, k)S(m, m-k)}{k\binom{m-1}{k}} \right) \cdot x_1 x_2$$

when $n < m$.

We want to compute the γ-dimension of $x = ax_1 + bx_2 + lx_1x_2$. If $l = 0$, the result is clear. We can now assume that $l \neq 0$. If l is different from $ab\,(n-1)!\,(m-1)!/(n+m-1)!$, we see that γ-dim$(x) = n+m$. On the other side, if l has precisely this value, then $\gamma^m(x) \neq 0$, because in this case $b \neq 0$, and by Proposition 10.1 we get γ-dim$(x) = m$ precisely. This gives another proof of Theorem 7.1.

Let us now pass to the "doubling formula".

THEOREM 11.1. *Let $q \leq n \leq m$ be positive integers; then*

$$S(m+n, m+q) = n\binom{n+m-1}{n} \sum_{k=q}^{n} \frac{S(n, k)S(m, m+q-k)}{k\binom{m+q-1}{k}}.$$

We called this a "doubling formula" because, particularizing to $n = m$, we get an expression allowing one to compute $S(2n, n+q)$ in terms of the numbers $S(n, k)$ with $q \leq k \leq n-1$.

Proof. This is an immediate consequence of Proposition 10.1 and the above computations. □

An alternative proof would be to invoke Theorem 7.1 rather than Proposition 10.1.

After trying to verify on a computer the analogous formula for Stirling numbers of the first kind, namely

$$s(n, k) = \sum_{j=0}^{n-k} \binom{-k}{n-k+j}\binom{2n-k}{n+j} S(n-k+j, j),$$

we were led to conjecture it:

THEOREM 11.2. *Let $q \leq n \leq m$ be positive integers; then*

$$s(m+n, m+q) = n\binom{n+m-1}{n} \sum_{k=q}^{n} \frac{s(n, k)s(m, m+q-k)}{k\binom{m+q-1}{k}}.$$

We call it a "theorem", since, after we had informed him about Theorem 11.1 and our conjecture, Al Lundell sent us a proof of the latter. The elegant proof is "elementary" in the following sense: it uses only some basic formulas for Stirling numbers (such as generating functions) and a contour argument in the computation of an integral, but no K-theory. Moreover, his proof encompasses the Stirling numbers of both the first *and* the second kind in a unified way.

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