

# 8. The Whitehead product and the positive cone

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## 8. THE WHITEHEAD PRODUCT AND THE POSITIVE CONE

We will establish an interesting connection between the positive cone of a product  $S^{2n} \times S^{2m}$  and the Whitehead product structure on the homotopy groups of the spaces  $BU(k)$ . As an application we will get some precise information on the positive cone of  $S^{2n} \times S^{2m}$ .

Let us first recall the basic properties of the Whitehead product (the reader may refer to [White]). The product  $S^p \times S^q$  has a cell structure obtained by attaching a  $(p+q)$ -cell to  $S^p \vee S^q$ . More precisely, there exists a suitable pointed map  $f_0: S^{p+q-1} \rightarrow S^p \vee S^q$  such that  $S^p \times S^q$  is homeomorphic to the mapping cone of  $f_0$ :

$$S^p \times S^q \cong C_{f_0} = (S^p \vee S^q) \cup_{f_0} e^{p+q}$$

Given a pointed map  $g = \alpha \vee \beta: S^p \vee S^q \rightarrow X$ , where  $X$  is a CW-complex, there exists (up to homotopy) an extension  $\bar{g}: S^p \times S^q \rightarrow X$  of  $g$  if and only if the composition  $g \circ f_0$  is homotopically trivial. Now, considering  $\alpha$  and  $\beta$  as elements of the homotopy groups  $\pi_p(X)$  and  $\pi_q(X)$  respectively, the composition  $(\alpha \vee \beta) \circ f_0$  determines an element in the homotopy group  $\pi_{p+q-1}(X)$ . This defines a map

$$\pi_p(X) \times \pi_q(X) \rightarrow \pi_{p+q-1}(X), (\alpha, \beta) \mapsto [\alpha, \beta] := (\alpha \vee \beta) \circ f_0,$$

which by definition is the Whitehead product. One can show that it is  $\mathbf{Z}$ -bilinear (provided that  $p, q \geq 2$ ), i.e.

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \quad \text{and} \quad [\alpha, \beta_1 + \beta_2] = [\alpha, \beta_1] + [\alpha, \beta_2].$$

Moreover, the Whitehead product is natural with respect to pointed maps, i.e. if  $f: X \rightarrow Y$  is a pointed map between CW-complexes, then

$$[f_*(\alpha), f_*(\beta)] = f_*([\alpha, \beta]).$$

We now want to study the case where  $X = BU(l)$ . Let  $x_1$  and  $x_2$  be two generators of  $\tilde{K}(S^{2n})$  and  $\tilde{K}(S^{2m})$  respectively, and assume  $1 \leq n \leq m$ . By Theorem 4.1, we know that  $\text{g-dim}(x_1) = n$  and that  $\text{g-dim}(x_2) = m$ . Letting  $q \geq m$ , we consider  $x_1$  and  $x_2$  as maps from  $S^{2n}$  (respectively  $S^{2m}$ ) to  $BU$  that lift to  $BU(q)$ . The element  $x_1 + x_2$  of  $\tilde{K}(S^{2n} \vee S^{2m}) = \tilde{K}(S^{2n}) \oplus \tilde{K}(S^{2m})$  can be represented by the map  $x_1 \vee x_2: S^{2n} \vee S^{2m} \rightarrow BU$ , and it also lifts to a map  $z: S^{2n} \vee S^{2m} \rightarrow BU(q)$ .

CLAIM. For  $k \in \{m, m + 1, \dots, m + n - 1\}$ , there is no extension of  $z = x_1 \vee x_2: S^{2n} \vee S^{2m} \longrightarrow BU(k)$  to a map  $S^{2n} \times S^{2m} \longrightarrow BU(k)$ .

Let  $y: S^{2n} \times S^{2m} \longrightarrow BU(s)$  be an extension of  $z$  for some  $s \geq m$ . Let  $x$  be the composition of  $y$  with the map  $i_s: BU(s) \longrightarrow BU$ . This means that  $\text{g-dim}(x) \leq s$  and that  $\iota^*(x) = x_1 + x_2 \in \tilde{K}(S^{2n} \vee S^{2m})$ , where  $\iota$  is the inclusion of  $S^{2n} \vee S^{2m}$  in the product  $S^{2n} \times S^{2m}$ . Recall that  $(\iota^*)^{-1}(x_1 + x_2) = x_1 + x_2 + \mathbf{Z} \cdot x_1x_2 \subset \tilde{K}(S^{2n} \times S^{2m})$ . So, there exists an integer  $l$  such that  $x = x_1 + x_2 + lx_1x_2$ , and consequently

$$\gamma^{n+m}(x) = (-1)^{n+m-1}(l(n+m-1)! - (n-1)!(m-1)!) \cdot x_1x_2 \neq 0.$$

We see that  $s \geq \text{g-dim}(x) \geq \gamma\text{-dim}(x) \geq n + m$ . This proves the claim.

As a direct consequence, by considering  $x_1$  and  $x_2$  as elements (in fact generators) of  $\pi_{2n}(BU(k))$  and  $\pi_{2m}(BU(k))$  respectively, we get the following result on the Whitehead product:

$$[x_1, x_2] \neq 0 \text{ in } \pi_{2n+2m-1}(BU(k)), \text{ for } m \leq k < n + m.$$

We would now like to get some information on the order of  $[x_1, x_2]$  in the homotopy group  $\pi_{2n+2m-1}(BU(k))$ . By  $\mathbf{Z}$ -bilinearity of the Whitehead product, we have  $ab[x_1, x_2] = [ax_1, bx_2]$  for any integers  $a$  and  $b$ . Replacing  $x_1$  by  $ax_1$  and  $x_2$  by  $bx_2$  in the preceding computation (in particular  $x = ax_1 + bx_2 + lx_1x_2$  for some  $l$ ), one easily verifies that

$$(*) \quad \left. \begin{array}{l} ab[x_1, x_2] = 0 \\ \text{in } \pi_{2n+2m-1}(BU(k)) \\ \text{for } m \leq k < n + m \end{array} \right\} \implies l(n+m-1)! - ab(n-1)!(m-1)! = 0$$

and this implies that  $ab$  is a multiple of  $(n+m-1)! / ((n-1)!(m-1)!)$ . Notice that  $[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$  has to be a torsion element. Indeed, by Lemma 4.2, the group  $\pi_{2n+2m-1}(BU(m))$  is finite, and the result follows from naturality of the Whitehead product. (In fact, one can show that any group  $\pi_{2i+1}(BU(j))$  is finite; this is proved like Lemma 4.2, by appealing to a result of Borel and Hirzebruch: see Remark i) in Section 9.) We have thus obtained the following theorem.

**THEOREM 8.1.** *Let  $1 \leq n \leq m$  and  $m \leq k < n + m$ . Let  $x_1$  and  $x_2$  be generators of the homotopy groups  $\pi_{2n}(BU(k)) \cong \mathbf{Z}$  and  $\pi_{2m}(BU(k)) \cong \mathbf{Z}$  respectively. Then the Whitehead product  $[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$  is non-zero. Moreover, its order is a multiple of  $\frac{(n+m-1)!}{(n-1)!(m-1)!}$ .*

By the implication (\*), if  $ab[x_1, x_2] = 0$  in  $\pi_{2n+2m-1}(BU(k))$  for some  $k$  with  $m \leq k < n + m$ , then for  $l = ab(n-1)!(m-1)!/(n+m-1)!$ , the geometric dimension of  $x := ax_1 + bx_2 + lx_1x_2$  is  $\leq k$  (and for any other value of  $l$ ,  $\text{g-dim}(x)$  is  $m+n$ , provided that  $ab \neq 0$ ). Surprisingly, this condition only depends on  $l$  and on the product  $ab$ . Consequently, from Theorem 2.3 together with Theorem 7.1, we obtain the following result.

**THEOREM 8.2.** *The geometric dimension on  $\tilde{K}(S^{2n} \times S^{2m})$ , with  $n \leq m$ , is given as follows: for  $x = ax_1 + bx_2 + lx_1x_2 \in \tilde{K}(S^{2n} \times S^{2m})$ ,*

$$\text{g-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ n & \text{if } a \neq 0, b = l = 0 \\ m & \text{if } a = 0, b \neq 0, l = 0 \\ s(ab) & \text{if } b \neq 0, l = ab(n-1)!(m-1)!/(n+m-1)! \\ n+m & \text{if } l \neq ab(n-1)!(m-1)!/(n+m-1)! \end{cases}$$

where  $s(ab) \in \{m, m+1, \dots, n+m-1\}$  only depends on the product  $ab$  (for fixed  $n$  and  $m$ ).

As a direct consequence of Theorems 8.1 and 8.2, we have

**COROLLARY 8.3.** *The order of the Whitehead product  $[x_1, x_2]$  in  $\pi_{2n+2m-1}(BU(n+m-1))$  is exactly  $(n+m-1)!/((n-1)!(m-1)!)$ .*

**REMARK 8.4.**

i) This result has been established only using information on the  $\gamma$ -cone of  $S^{2n} \times S^{2m}$  (and Serre's theorem on the rational homotopy of spheres). If one is able to compute its positive cone, one then can easily compute the exact order of  $[x_1, x_2]$  in the various homotopy groups  $\pi_{2n+2m-1}(BU(k))$ , for  $m \leq k < n+m$ : it is given by

$$\min \left\{ l \geq 1 \mid \text{g-dim} \left( l \frac{(n+m-1)!}{(n-1)!(m-1)!} x_1 + x_2 + lx_1x_2 \right) \leq k \right\}.$$

ii) In 1960, Bott [Bott3] has proved Corollary 8.3 by different methods.