8. The Whitehead product and the positive cone

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **47 (2001)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.08.2024**

8. The Whitehead product and the positive cone

We will establish an interesting connection between the positive cone of a product $S^{2n} \times S^{2m}$ and the Whitehead product structure on the homotopy groups of the spaces $BU(k)$. As an application we will get some precise information on the positive cone of $S^{2n} \times S^{2m}$.

Let us first recall the basic properties of the Whitehead product (the reader may refer to [White]). The product $S^p \times S^q$ has a cell structure obtained by attaching a $(p+q)$-cell to $S^p \vee S^q$. More precisely, there exists a suitable pointed map $f_0 \colon S^{p+q-1} \longrightarrow S^p \vee S^q$ such that $S^p \times S^q$ is homeomorphic to the mapping cone of f_0:

$$S^p \times S^q \cong C_{f_0} = (S^p \vee S^q) \cup_{f_0} e^{p+q}$$

Given a pointed map $g = \alpha \vee \beta \colon S^p \vee S^q \longrightarrow X$, where X is a CW-complex, there exists (up to homotopy) an extension $\bar{g} \colon S^p \times S^q \longrightarrow X$ of g if and only if the composition $g \circ f_0$ is homotopically trivial. Now, considering α and β as elements of the homotopy groups $\pi_p(X)$ and $\pi_q(X)$ respectively, the composition $(\alpha \vee \beta) \circ f_0$ determines an element in the homotopy group $\pi_{p+q-1}(X)$. This defines a map

$$\pi_p(X) \times \pi_q(X) \longrightarrow \pi_{p+q-1}(X)\,, \ (\alpha, \beta) \longmapsto [\alpha, \beta] := (\alpha \vee \beta) \circ f_0\,,$$

which by definition is the Whitehead product. One can show that it is $\mathbf{Z}$-bilinear (provided that $p, q \geq 2$), i.e.

$$[\alpha_1 + \alpha_2, \beta] = [\alpha_1, \beta] + [\alpha_2, \beta] \quad \text{and} \quad [\alpha, \beta_1 + \beta_2] = [\alpha, \beta_1] + [\alpha, \beta_2]\,.$$

Moreover, the Whitehead product is natural with respect to pointed maps, i.e. if $f \colon X \longrightarrow Y$ is a pointed map between CW-complexes, then

$$[f_*(\alpha), f_*(\beta)] = f_*([\alpha, \beta])\,.$$

We now want to study the case where $X = BU(l)$. Let x_1 and x_2 be two generators of $\widetilde{K}(S^{2n})$ and $\widetilde{K}(S^{2m})$ respectively, and assume $1 \leq n \leq m$. By Theorem 4.1, we know that g-dim$(x_1) = n$ and that g-dim$(x_2) = m$. Letting $q \geq m$, we consider x_1 and x_2 as maps from S^{2n} (respectively S^{2m}) to BU that lift to $BU(q)$. The element $x_1 + x_2$ of $\widetilde{K}(S^{2n} \vee S^{2m}) = \widetilde{K}(S^{2n}) \oplus \widetilde{K}(S^{2m})$ can be represented by the map $x_1 \vee x_2 \colon S^{2n} \vee S^{2m} \longrightarrow BU$, and it also lifts to a map $z \colon S^{2n} \vee S^{2m} \longrightarrow BU(q)$.

CLAIM. *For* $k \in \{m, m+1, \dots, m+n-1\}$, *there is no extension of* $z = x_1 \vee x_2 \colon S^{2n} \vee S^{2m} \longrightarrow BU(k)$ *to a map* $S^{2n} \times S^{2m} \longrightarrow BU(k)$.

Let $y \colon S^{2n} \times S^{2m} \longrightarrow BU(s)$ be an extension of z for some $s \geq m$. Let x be the composition of y with the map $i_s \colon BU(s) \longrightarrow BU$. This means that g-dim$(x) \leq s$ and that $\iota^*(x) = x_1 + x_2 \in \widetilde{K}(S^{2n} \vee S^{2m})$, where ι is the inclusion of $S^{2n} \vee S^{2m}$ in the product $S^{2n} \times S^{2m}$. Recall that $(\iota^*)^{-1}(x_1 + x_2) = x_1 + x_2 + \mathbf{Z} \cdot x_1 x_2 \subset \widetilde{K}(S^{2n} \times S^{2m})$. So, there exists an integer l such that $x = x_1 + x_2 + l x_1 x_2$, and consequently

$$\gamma^{n+m}(x) = (-1)^{n+m-1}(l\,(n+m-1)! - (n-1)!\,(m-1)!) \cdot x_1 x_2 \neq 0\,.$$

We see that $s \geq$ g-dim$(x) \geq \gamma$-dim$(x) \geq n+m$. This proves the claim.

As a direct consequence, by considering x_1 and x_2 as elements (in fact generators) of $\pi_{2n}(BU(k))$ and $\pi_{2m}(BU(k))$ respectively, we get the following result on the Whitehead product:

$$[x_1, x_2] \neq 0 \text{ in } \pi_{2n+2m-1}(BU(k))\,, \quad \text{for } m \leq k < n+m\,.$$

We would now like to get some information on the order of $[x_1, x_2]$ in the homotopy group $\pi_{2n+2m-1}(BU(k))$. By $\mathbf{Z}$-bilinearity of the Whitehead product, we have $ab[x_1, x_2] = [ax_1, bx_2]$ for any integers a and b. Replacing x_1 by ax_1 and x_2 by bx_2 in the preceding computation (in particular $x = ax_1 + bx_2 + l x_1 x_2$ for some l), one easily verifies that

$$(*) \qquad \left.\begin{array}{l} ab[x_1, x_2] = 0 \\ \text{in } \pi_{2n+2m-1}(BU(k)) \\ \text{for } m \leq k < n+m \end{array}\right\} \Longrightarrow l\,(n+m-1)! - ab(n-1)!\,(m-1)! = 0$$

and this implies that ab is a multiple of $(n+m-1)!/((n-1)!\,(m-1)!)$. Notice that $[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$ has to be a torsion element. Indeed, by Lemma 4.2, the group $\pi_{2n+2m-1}(BU(m))$ is finite, and the result follows from naturality of the Whitehead product. (In fact, one can show that any group $\pi_{2i+1}(BU(j))$ is finite; this is proved like Lemma 4.2, by appealing to a result of Borel and Hirzebruch: see Remark i) in Section 9.) We have thus obtained the following theorem.

THEOREM 8.1. *Let* $1 \leq n \leq m$ *and* $m \leq k < n+m$. *Let* x_1 *and* x_2 *be generators of the homotopy groups* $\pi_{2n}(BU(k)) \cong \mathbf{Z}$ *and* $\pi_{2m}(BU(k)) \cong \mathbf{Z}$ *respectively. Then the Whitehead product* $[x_1, x_2] \in \pi_{2n+2m-1}(BU(k))$ *is non-zero. Moreover, its order is a multiple of* $\dfrac{(n+m-1)!}{(n-1)!\,(m-1)!}$.

By the implication (*), if $ab[x_1, x_2] = 0$ in $\pi_{2n+2m-1}(BU(k))$ for some k with $m \leq k < n+m$, then for $l = ab\,(n-1)!\,(m-1)!/(n+m-1)!$, the geometric dimension of $x := ax_1 + bx_2 + lx_1x_2$ is $\leq k$ (and for any other value of l, g-dim(x) is $m+n$, provided that $ab \neq 0$). Surprisingly, this condition only depends on l and on the product ab. Consequently, from Theorem 2.3 together with Theorem 7.1, we obtain the following result.

THEOREM 8.2. *The geometric dimension on $\widetilde{K}(S^{2n} \times S^{2m})$, with $n \leq m$, is given as follows: for $x = ax_1 + bx_2 + lx_1x_2 \in \widetilde{K}(S^{2n} \times S^{2m})$,*

$$\text{g-dim}(x) = \begin{cases} 0 & \text{if } a = b = l = 0 \\ n & \text{if } a \neq 0,\ b = l = 0 \\ m & \text{if } a = 0,\ b \neq 0,\ l = 0 \\ s(ab) & \text{if } b \neq 0,\ l = ab\,(n-1)!\,(m-1)!/(n+m-1)! \\ n+m & \text{if } l \neq ab\,(n-1)!\,(m-1)!/(n+m-1)! \end{cases}$$

where $s(ab) \in \{m, m+1, \ldots, n+m-1\}$ only depends on the product ab (for fixed n and m).

As a direct consequence of Theorems 8.1 and 8.2, we have

COROLLARY 8.3. *The order of the Whitehead product $[x_1, x_2]$ in $\pi_{2n+2m-1}(BU(n+m-1))$ is exactly $(n+m-1)!/((n-1)!\,(m-1)!)$.*

REMARK 8.4.

i) This result has been established only using information on the γ-cone of $S^{2n} \times S^{2m}$ (and Serre's theorem on the rational homotopy of spheres). If one is able to compute its positive cone, one then can easily compute the exact order of $[x_1, x_2]$ in the various homotopy groups $\pi_{2n+2m-1}(BU(k))$, for $m \leq k < n+m$: it is given by

$$\min\left\{ l \geq 1 \;\middle|\; \text{g-dim}\left(l\frac{(n+m-1)!}{(n-1)!\,(m-1)!}\,x_1 + x_2 + lx_1x_2 \right) \leq k \right\}.$$

ii) In 1960, Bott [Bott3] has proved Corollary 8.3 by different methods.