

§4. Numerical results

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 1-2: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.08.2024**

Nutzungsbedingungen

Die ETH-Bibliothek ist Anbieterin der digitalisierten Zeitschriften. Sie besitzt keine Urheberrechte an den Inhalten der Zeitschriften. Die Rechte liegen in der Regel bei den Herausgebern.

Die auf der Plattform e-periodica veröffentlichten Dokumente stehen für nicht-kommerzielle Zwecke in Lehre und Forschung sowie für die private Nutzung frei zur Verfügung. Einzelne Dateien oder Ausdrucke aus diesem Angebot können zusammen mit diesen Nutzungsbedingungen und den korrekten Herkunftsbezeichnungen weitergegeben werden.

Das Veröffentlichen von Bildern in Print- und Online-Publikationen ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Die systematische Speicherung von Teilen des elektronischen Angebots auf anderen Servern bedarf ebenfalls des schriftlichen Einverständnisses der Rechteinhaber.

Haftungsausschluss

Alle Angaben erfolgen ohne Gewähr für Vollständigkeit oder Richtigkeit. Es wird keine Haftung übernommen für Schäden durch die Verwendung von Informationen aus diesem Online-Angebot oder durch das Fehlen von Informationen. Dies gilt auch für Inhalte Dritter, die über dieses Angebot zugänglich sind.

TABLE 2

q	32	128	512	2048	8192
interval	[0, 6]	[0, 12]	[10, 34]	[64, 108]	[300, 384]

§4. NUMERICAL RESULTS

In order to obtain numerical results on $N(A, B)$ to test our heuristics the first remark is that $N(A_1, B) = N(A_2, B)$ if $\text{Tr}(A_1) = \text{Tr}(A_2)$. So we have to distinguish only between $\text{Tr}(A) = 0$ and $\text{Tr}(A) = 1$. We shall compute the trace of Frobenius for the seven factors of our Jacobian. We shall write $f_4 = f_1 + f_2$, $f_5 = f_1 + f_3$, $f_6 = f_2 + f_3$ and $f_7 = f_1 + f_2 + f_3$. The Jacobians of the curves C_{f_i} given by $y^2 + y = f_i$ for $i = 1, \dots, 7$ constitute the seven factors of $\text{Jac}(C_{A,B})$. We write

$$n_{f_i} = \#\{x \in \mathbf{F}_q^* : \text{Tr}(f_i(x)) = 0\}.$$

(4.1) PROPOSITION. *The number of solutions $N(A, B)$ over $\mathbf{F}_{q=2^m}$ with m odd of the system (1) with $\lambda = A^2 + A + 1 + B \neq 0$ is given by*

$$N(A, B) = \frac{2q - 2 - 2(n_{f_1} + n_{f_2} + n_{f_3} - n_{f_4} - n_{f_5} - n_{f_6} + n_{f_7})}{24} \quad \text{if } \text{Tr}(A) = 0,$$

and

$$N(A, B) = \frac{-6q - 2 + 2 \sum_{i=1}^7 n_{f_i}}{24} \quad \text{if } \text{Tr}(A) = 1.$$

Proof. As just explained we may take $A = 0$ or $A = 1$. Then $\lambda = B + 1 \neq 0$ and we set $f_1 = (B + 1)(x^3 + x)$, $f_2 = (B + 1)(1/x^3 + 1/x)$ and $f_3 = (B + 1)(x + 1/x)$. Then $C_{1,B} = C_{f_1} \times_{\mathbf{P}^1} C_{f_2} \times_{\mathbf{P}^1} C_{f_3}$ and $C_{0,B} = C_{f_1+1} \times_{\mathbf{P}^1} C_{f_2+1} \times_{\mathbf{P}^1} C_{f_3+1}$. As in Theorem (2.3) the curves C_{f_i} for $i = 4, \dots, 7$ give the remaining traces of Frobenius.

The trace of Frobenius $t(C_{f_i})$ is of the form

$$t(C_{f_i}) = q + 1 - 2n_{f_i} - a_i,$$

where a_i is the contribution of $x = 0, \infty$, while the trace of Frobenius of C_{f_i+1} is

$$t(C_{f_i}) = -q + 3 + 2n_{f_i} - b_i,$$

where b_i is the contribution of $x = 0, \infty$. By analyzing these contributions from 0 and ∞ one gets the proposition.

We now give tables with the distribution of the numbers $N(A, B)$ for $q = 2^m$ with m odd and $5 \leq m \leq 13$. These tables are obtained by computing the numbers n_{f_i} and they solve the coset weight distribution problem for the corresponding $BCH(3)$ codes. The first unknown case up to now was $q = 2^9$, see [C-Z]. Moreover, the tables confirm our heuristics. We list the frequencies divided by $q/2$.

TABLE 3

$q = 2^5 :$

$N(A, B)$	0	2
frequency	27	35

$q = 2^7 :$

$N(A, B)$	0	2	4	6	8	10
frequency	2	28	98	84	35	7

$q = 2^9 :$

$N(A, B)$	12	14	16	18	20	22	24	26	28	30	32
frequency	18	21	117	180	148	195	199	81	36	18	9

$q = 2^{11} :$

$N(A, B)$	66	68	70	72	74	76	78	80	82	84	86
frequency	22	66	88	55	176	264	187	374	374	374	451
$N(A, B)$	88	90	92	94	96	98	100	102	104	106	108
frequency	365	341	275	341	154	44	55	33	11	22	22

$q = 2^{13} :$

In this case we encounter a new phenomenon. The function $N(A, B)$ assumes even values in the interval $[290, 390]$, but not all even values are taken. This contradicts the expectation of [C-Z] that the values form a sequence

of even integers without gaps. The frequency divided by $q/2$ of the value $290 + 2\ell$ with $0 \leq \ell \leq 50$ is given by

$$13 \gamma_\ell + \begin{cases} 1 & \text{if } \ell = 11, \\ 1 & \text{if } \ell = 37, \\ 0 & \text{else,} \end{cases}$$

where $\gamma = (\gamma_0, \dots, \gamma_{50})$ is the vector

$$\gamma = (1, 0, 1, 0, 1, 0, 6, 3, 5, 5, 12, 7, 19, 15, 22, 25, 37, 40, 43, 37, 35, 60, 54, 72, 72, 58, 65, 61, 57, 57, 63, 48, 35, 44, 34, 34, 25, 29, 25, 15, 9, 7, 2, 3, 7, 3, 3, 1, 0, 1, 2).$$

In accordance with our heuristics less than 1% of the $N(A, B)$ lie outside the interval $[300, 384]$.

§5. THE COVERING RADIUS

A problem in coding theory that precedes the coset weight distribution problem is the determination of the covering radius. It is defined for a binary linear code \mathcal{C} of length n as the smallest integer ρ such that the spheres of radius ρ around the codewords cover \mathbf{F}_2^n . Equivalently, it is the maximum weight of a coset leader (by which we mean a vector of minimum weight in a coset of \mathcal{C} in \mathbf{F}_2^n). It is an interesting parameter of a code since it provides information on the performance of the code when used in data compression.

In a series of papers [H-B], [A-M] and [H], of which [H-B] and [H] treat the case m even and [A-M] the case m odd, it was proved that the $BCH(3)$ code of length $n = 2^m - 1$ has covering radius

$$\rho(BCH(3)) = 5 \quad \text{for } m \geq 4.$$

The proofs for the various cases are very different. Using algebraic geometry we can give a unified proof.

In order to prove that $\rho(BCH(3)) = 5$ we have to show that for every $(A, B, C) \in \mathbf{F}_q^3$ the system of equations:

$$(15) \quad \begin{aligned} x_1 + \dots + x_5 &= A, \\ x_1^3 + \dots + x_5^3 &= B, \\ x_1^5 + \dots + x_5^5 &= C, \end{aligned}$$

has a solution $(x_1, \dots, x_5) \in \mathbf{F}_q^5$. On replacing x_i by $x_i + A$ we may assume without loss of generality that $A = 0$ and $(B, C) \neq (0, 0)$. If we then