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ON THE RATIONAL FORMS OF NILPOTENT LIE ALGEBRAS AND LATTICES IN NILPOTENT LIE GROUPS

by Yu. S. SEMENOV *)

ABSTRACT. We study the rational forms of real finite-dimensional nilpotent Lie algebras and the corresponding lattices in nilpotent Lie groups. We show that for some Lie algebras there are infinitely many such rational forms up to isomorphism and give a description of isomorphism classes in several 6-dimensional cases. Nilpotent Lie algebras with a unique rational form are also considered.

1. INTRODUCTION

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbf{R}$ and $\mathfrak{h}$ be a $\mathbf{Q}$-subalgebra of $\mathfrak{g}$. We say that $\mathfrak{h}$ is a rational form (or $\mathbf{Q}$-form) of $\mathfrak{g}$ if there exists a basis X of $\mathfrak{h}$ over $\mathbf{Q}$ such that X is a basis of $\mathfrak{g}$ over $\mathbf{R}$. In other words, the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ gives rise to an isomorphism $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{g}$.

In the sequel all Lie algebras are assumed to be nilpotent and finite-dimensional unless otherwise specified. The main purpose of the present work is to describe rational forms for some real nilpotent Lie algebras. The rational forms (or their isomorphism classes) in such algebras are closely related to lattices, i.e., discrete cocompact subgroups in nilpotent Lie groups.

Let G be a nilpotent connected 1-connected Lie group and $\mathfrak{g}$ be the Lie algebra of G. It is well known that $\exp\colon \mathfrak{g} \to G$ and $\log\colon G \to \mathfrak{g}$ are two reciprocal diffeomorphisms. Let $\mathfrak{h}$ be a rational form of $\mathfrak{g}$ and $X = \{x_1, \dots, x_d\}$ be a basis of $\mathfrak{h}$. Malcev showed in [5] that the subgroup Γ of G generated by $\exp(rx_1), \dots, \exp(rx_d)$ (where r is an appropriate integer) is a lattice of G.

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If we change the basis of $\mathfrak{h}$ we can obtain another subgroup Γ_1 which is strictly commensurable to Γ. Here are three different definitions of commensurability of lattices.

DEFINITION 1. Two lattices Γ_1 and Γ_2 are strictly commensurable in a Lie group G if $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in both Γ_1 and Γ_2.

DEFINITION 2. Two lattices Γ_1 et Γ_2 are *commensurable* in a Lie group G if there is an element $g \in G$ such that Γ_1 and $g^{-1}\Gamma_2 g$ are strictly commensurable in G.

DEFINITION 3. Two lattices Γ_1 and Γ_2 are abstractly commensurable if there are two subgroups of finite index $H_i \leq \Gamma_i$ $(i = 1, 2)$ and an isomorphism $H_1 \cong H_2$.

We can deduce the following proposition from Malcev's results.

PROPOSITION 1.1. *There are three bijections:*

1. $$\{\mathbf{Q}\text{-}\mathit{forms\ of}\ \mathfrak{g}\} \cong \left\{\begin{matrix} \mathit{Lattices}\ \Gamma\ \mathit{of}\ G \\ \mathit{up\ to\ strict\ commensurability} \end{matrix}\right\};$$

2. $$\left\{\begin{matrix} \mathbf{Q}\text{-}\mathit{forms\ of}\ \mathfrak{g} \\ \mathit{up\ to\ adjoint\ automorphism}\ \mathrm{Ad}(g) \end{matrix}\right\} \cong \left\{\begin{matrix} \mathit{Lattices}\ \Gamma\ \mathit{of}\ G \\ \mathit{up\ to\ commensurability} \end{matrix}\right\};$$

3. $$\left\{\begin{matrix} \mathbf{Q}\text{-}\mathit{forms\ of}\ \mathfrak{g} \\ \mathit{up\ to}\ \mathbf{Q}\text{-}\mathit{isomorphism} \end{matrix}\right\} \cong \left\{\begin{matrix} \mathit{Lattices}\ \Gamma\ \mathit{of}\ G \\ \mathit{up\ to\ abstract\ commensurablity} \end{matrix}\right\}.$$

Proof. The bijection 1 is a classical result of Malcev [5]. The bijection 2 follows immediately from the first one and the fact that the $\mathbf{Q}$-form $\mathrm{Ad}(g)\mathfrak{h}$ corresponds to the lattice $g\Gamma g^{-1}$ if and only if $\mathfrak{h}$ corresponds to Γ.

Let us prove the existence of the bijection 3. Let Γ_1 be a lattice abstractly commensurable with Γ_2 and $\mathfrak{h}_1$, $\mathfrak{h}_2$ be the corresponding $\mathbf{Q}$-forms. We may assume from the very beginning (possibly passing to the subgroups of finite index) that there is an isomorphism $\alpha\colon \Gamma_1 \to \Gamma_2$. It is known [5] that α can be extended to an automorphism $\hat{\alpha}\colon G \to G$ and this automorphism gives rise to the automorphism $d\hat{\alpha}\colon \mathfrak{g} \to \mathfrak{g}$ of the Lie algebra $\mathfrak{g}$. It is easy to see that $d\hat{\alpha}(\mathfrak{h}_1) = \mathfrak{h}_2$.

Conversely, let $\beta\colon \mathfrak{h}_1 \to \mathfrak{h}_2$ be an isomorphism of Lie algebras over $\mathbf{Q}$. It is clear that β can be considered as an $\mathbf{R}$-automorphism $\mathfrak{g} \to \mathfrak{g}$. It induces the automorphism $B\colon G \to G$ by means of the exponential map. Let $\Gamma_2 = B(\Gamma_1)$.

Then $\Gamma_1 \cong \Gamma_2$. Moreover, the $\mathbf{Q}$-form $\mathfrak{h}_2$ corresponds to the lattice Γ_2. Indeed, if $\exp(x) \in \Gamma_2$ then $\exp(x) = B(\exp(y)) = \exp(\beta(y))$ for some $y \in \mathfrak{h}_1$. Finally, $x = \beta(y) \in \mathfrak{h}_2$. □

In this paper we are mostly interested in nilpotent Lie algebras with many rational forms (up to isomorphism). However, there are examples of nilpotent Lie algebras without rational forms. One of them, involving a Lie algebra of dimension 7 and class 6, is due to N. Bourbaki (see [1, Chap.1, §4, ex.18]). In fact, this is the minimal dimension for such an example. There are nilpotent Lie algebras of class 2 and of dimension $d \geq 10$ (see [5, 4]) without rational forms. Note that the corresponding Lie groups have no lattices at all.

It is a trivial exercise to show that every abelian Lie algebra has a unique rational form up to isomorphism. It follows from the results of Dixmier [3] that every real nilpotent Lie algebra $\mathfrak{g}$ of dimension ≤ 5 has the same property.

Consider the following *central* extension over $\mathbf{R}$ of an abelian finite dimensional real Lie algebra $\mathfrak{a}$:

$$0 \to \mathfrak{c} \to \mathfrak{b} \to \mathfrak{a} \to 0 \tag{1.1}$$

where $\mathfrak{c}$ is supposed to be a 1-dimensional ideal. The algebra $\mathfrak{b}$ is in fact the direct sum of a generalized Heisenberg algebra and an abelian one. In Section 2 we show that all such $\mathfrak{b}$ have a unique rational form (up to isomorphism) as well as all *free* real nilpotent Lie algebras.

In Section 3 we consider Malcev's example of a 6-dimensional Lie algebra of class 3 having infinitely many non-isomorphic rational forms in more details. Let $t \in \mathbf{R}$. Consider the nilpotent Lie algebra $\mathfrak{g}_t$ with a basis $x_1, \ldots, x_6$ and the structure of Lie algebra given by the following relations:

$$\begin{gathered}[x_1,x_2] = x_4\,, \quad [x_1,x_3] = x_6\,, \quad [x_1,x_4] = x_5\,, \\ [x_2,x_3] = x_5 + tx_6\,, \quad [x_2,x_4] = x_6\,,\end{gathered} \tag{1.2}$$

other brackets being trivial. Malcev showed [5] that for all $t \in \mathbf{R}$ there is an isomorphism $\mathfrak{g}_t \cong \mathfrak{g}_0$ over $\mathbf{R}$ but, for instance, $\mathfrak{g}_1$ and $\mathfrak{g}_0$ are not isomorphic over $\mathbf{Q}$.

The following theorem is proved.

THEOREM 1. *Let $s, t \in \mathbf{Q}$. The Lie algebras $\mathfrak{g}_s$ and $\mathfrak{g}_t$ are isomorphic over $\mathbf{Q}$ if and only if there is $q \in \mathbf{Q}$ such that $(s^2+4)(t^2+4) = q^2$.*

In Section 4 we propose a construction providing nilpotent Lie algebras with several non-isomorphic rational forms. As an application we prove the following theorem.

THEOREM 2. *Let* $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ *be a free nilpotent Lie algebra of class* $c \geq 2$ *on* p *generators. Then* $\mathfrak{g} \oplus \mathfrak{g}$ *and* $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ *(regarded over* $\mathbf{R}$*) have infinitely many non-isomorphic rational forms.*

In Theorem 3 we also classify all rational forms for three 6-dimensional real nilpotent Lie algebras $\mathfrak{g}$ (two of them appear in Theorem 2 for $p = c = 2$) which are of class 2 and have 2-dimensional centre coinciding with the derived subalgebra.

In conclusion let us mention a direct way to prove that two given lattices in a nilpotent Lie group are not commensurable. For example, let $G = UT_3(\mathbf{R})$ be the Lie group of upper triangular 3×3-matrices with 1 on the diagonal, $\mathfrak{g} = \mathfrak{f}_2(2, \mathbf{R})$ be Lie algebra of G. Consider $G \times G$ and its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ which has infinitely many non-isomorphic rational forms $\mathfrak{h}_m$ ($m \geq 1$ is a square-free integer), in view and in the notation of Theorems 2, 3 (see Section 4 for more details).

Let Γ_m and Γ_n be corresponding lattices in $G \times G$ for distinct m, n. One can prove that the ratio of the covolumes of Γ_m and Γ_n with respect to a Haar measure on $G \times G$ equals $m\sqrt{m}/n\sqrt{n}$ up to a rational factor. Hence the lattices are not commensurable. Note that by Proposition 1.1 and Theorem 3 Γ_m and Γ_n are not commensurable in any sense.

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2. NILPOTENT LIE ALGEBRAS WITH A UNIQUE RATIONAL FORM UP TO ISOMORPHISM

2.1 HEISENBERG ALGEBRAS

Let us begin with the following considerations that we will use here and in the next sections (see [2, Chapter 5] for more details). Suppose that a real Lie algebra $\mathfrak{g}$ has a $\mathbf{Q}$-form $\mathfrak{h}$ and $\mathfrak{i}$ (resp. $\mathfrak{a}$) is an ideal (resp. a subalgebra) of $\mathfrak{g}$. We say that $\mathfrak{i}$ (resp. $\mathfrak{a}$) is rational if $\mathfrak{i} \cap \mathfrak{h}$ (resp. $\mathfrak{a} \cap \mathfrak{h}$) is a rational form of $\mathfrak{i}$ (resp. $\mathfrak{a}$). For instance, the terms $C^k\mathfrak{g}$ of the lower central series of $\mathfrak{g}$ are rational as well as centralizers of rational subalgebras or ideals. It is not hard to see that $\mathfrak{h}/\mathfrak{i} \cap \mathfrak{h}$ is a rational form of the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$.

Let

$$\mathfrak{g} = \mathfrak{i}_1 > \mathfrak{i}_2 > \cdots > \mathfrak{i}_{k+1} = 0 \tag{2.1}$$

be a descending series of rational ideals of $\mathfrak{g}$. We say that a basis $X = \{x_1, \dots, x_d\}$ of a rational form $\mathfrak{h}$ is based on (2.1) if $x_1, \dots, x_{p_1}$ generate $\mathfrak{g} \bmod \mathfrak{i}_2$, $x_1, \dots, x_{p_2}$ generate $\mathfrak{g} \bmod \mathfrak{i}_3$ and so on. It can be shown that such a basis exists for any series (2.1). In the sequel we will use these kinds of bases for a suitable descending series dealing, for instance, with Heisenberg algebras.

Recall that the (generalized) Heisenberg algebra $\mathfrak{hei}_{2k+1}(\mathbf{R})$ has an $\mathbf{R}$-basis $H_1, \dots, H_{2k+1}$ in which

$$[H_1, H_2] = [H_3, H_4] = \cdots = [H_{2k-1}, H_{2k}] = H_{2k+1}, \tag{2.2}$$

other brackets being trivial. Here the 1-dimensional centre is spanned by H_{2k+1}.

Given an extension (1.1) one can attach to it a 2-cocycle $\omega\colon \Lambda^2\mathfrak{a} \to \mathbf{R}$ in the usual way. Also ω can be regarded as a symplectic form on $\mathfrak{a}$. If $\mathfrak{b} = \mathfrak{hei}_{2k+1}(\mathbf{R})$ then ω is the canonical non-degenerate symplectic form with respect to the basis $H_1, \dots, H_{2k}$ (mod $\mathbf{R} \cdot H_{2k+1}$).

Let $d = \dim_{\mathbf{R}} \mathfrak{a}$ and let $m = d - rank(\omega)$ be the codimension of the kernel of ω. It is not hard to see (cf. the proof of the proposition below) that the Lie algebra $\mathfrak{b}$ is uniquely defined up to $\mathbf{R}$-isomorphism by d and m. Namely,

$$\mathfrak{b} \cong \mathfrak{hei}_{d+1-m}(\mathbf{R}) \oplus \mathbf{R}^m .$$

This implies that the centre of $\mathfrak{b}$ is $(m+1)$-dimensional. Thus, two Lie algebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$ ($\dim_{\mathbf{R}} \mathfrak{b}_1 = \dim_{\mathbf{R}} \mathfrak{b}_2$) of type (1.1) are not isomorphic if $m_1 \neq m_2$.

Evidently, $\mathfrak{hei}_{2k+1}(\mathbf{Q})$ is a rational form for $\mathfrak{hei}_{2k+1}(\mathbf{R})$.

The following proposition holds.

PROPOSITION 2.1. *In the above notation let* $\mathfrak{h}$ *be a rational form of* $\mathfrak{b}$. *Let* $d = \dim(\mathfrak{b}) - 1$, $m = \dim[\mathfrak{b}, \mathfrak{b}]$ *and let* $\mathbf{Q}^m$ *denote the abelian Lie* $\mathbf{Q}$*-algebra of dimension* m. *Then*

$$\mathfrak{h} \cong \mathfrak{hei}_{d+1-m}(\mathbf{Q}) \oplus \mathbf{Q}^m$$

over $\mathbf{Q}$, *i.e., there is a unique rational form for* $\mathfrak{b}$ *up to isomorphism.*

Proof. Choose a $\mathbf{Q}$-basis $B_1, \dots, B_{d+1}$ for $\mathfrak{h}$. Either all brackets $[B_i, B_j] = 0$, and then $\mathfrak{h} \cong \mathbf{Q}^{d+1}$, or there are i,j such that $[B_i, B_j] = C \neq 0$.

We may suppose that $C = B_{d+1}$. Thus the derived subalgebra of $\mathfrak{h}$ is spanned by B_{d+1}. The corresponding symplectic form ω is represented by a skew-symmetric $d \times d$ matrix $M = (\mu_{ij})$ with respect to the basis $B_1, \dots, B_d$ (mod $[\mathfrak{h}, \mathfrak{h}]$). Namely, $[B_i, B_j] = \mu_{ij} B_{d+1}$. Over $\mathbf{Q}$ one can choose a canonical symplectic basis $\widehat{B_1}, \dots, \widehat{B_d}$ (mod $[\mathfrak{h}, \mathfrak{h}]$) so that the matrix $\widehat{M}$ representing ω has l blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of ω is equal to $2l$ and $2l = d - m$. In the basis $B_1, \dots, B_{d+1}$ (we omit the 'hats') of $\mathfrak{h}$

$$[B_1, B_2] = [B_3, B_4] = \cdots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

2.2 Example of a free nilpotent algebra

Let $\mathfrak{f}_c(n, \mathbf{R})$ be the free nilpotent Lie algebra of class c on n generators. Then $\mathfrak{f}_c(n, \mathbf{R})$ has a unique rational form $\mathfrak{f}_c(n, \mathbf{Q})$ up to isomorphism (cf. Theorem 2).

Indeed, let $\mathfrak{h} = \langle x_1, \dots, x_n, \dots \rangle$ be a rational form of $\mathfrak{f}_c(n, \mathbf{R})$. We may suppose that $x_1, \dots, x_n$ span (modulo the derived subalgebra) $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$. Consequently, $\mathfrak{h}$ is generated by $\{x_1, \dots, x_n\}$ as a Lie algebra. There exists an epimorphism $\pi\colon \mathfrak{f}_c(n, \mathbf{Q}) \to \mathfrak{h}$ because $\mathfrak{f}_c(n, \mathbf{Q})$ is free. It must be an isomorphism since the dimension of $\mathfrak{h}$ equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class c on n generators.

2.3 More examples

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let $\mathfrak{g}_t$, $t \in \mathbf{R}$, be a family of real 6-dimensional Lie algebras with a basis $\{x_1, \dots, x_6\}$ such that

$$[x_1, x_2] = x_3\,, \quad [x_1, x_3] = t x_5\,, \quad [x_1, x_5] = x_6\,,$$
$$[x_2, x_3] = x_4\,, \quad [x_2, x_4] = x_5\,, \quad [x_3, x_4] = x_6\,,$$

other brackets being trivial. One can show that

1. $C^k \mathfrak{g}_t = \langle x_{k+1}, \dots, x_6 \rangle$, $k = 2, \dots, 5$, where $C^k \mathfrak{g}_t$ are the terms of the lower central series of $\mathfrak{g}_t$.
2. The centralizer $\mathfrak{C}$ of $C^4 \mathfrak{g}_t$, that is, $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$ is spanned by $x_2, \dots, x_6$.

3. Real Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are not isomorphic but $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$.
4. If $t \in \mathbf{Q} \setminus \{0\}$ then the rational algebra $\mathfrak{g}_t \cong \mathfrak{g}_1$ over $\mathbf{Q}$.
5. $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are two Lie algebras with a unique rational form up to isomorphism.
6. Let $\mathfrak{g}$ be a split real simple Lie algebra of type G_2, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then $\mathfrak{n}_+$ is isomorphic to $\mathfrak{g}_0$.

3. Malcev's example

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a $\mathbf{Q}$-isomorphism between $\mathfrak{g}_t$ and $\mathfrak{g}_s$. It must be written in the following form (cf. [5]) since $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$, $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$ and the centralizer $\mathfrak{c}$ of $C^2\mathfrak{g}$, which is an ideal in this case, is spanned by $x_3, \ldots, x_6$.

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \ldots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \ldots \\ y_3 = \qquad\qquad\qquad\quad a_{33}x_3 + a_{34}x_4 + \ldots \\ y_4 = \qquad\qquad\qquad\qquad\qquad\quad a_{44}x_4 + \ldots \end{cases}$$

We do not explicit the expressions for y_5, y_6. Here $y_1, \ldots, y_6$ are basis elements of $\mathfrak{g}_s$ satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \ldots ,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$. On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

$$\begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases} \tag{3.1}$$

We need to compute the remaining two brackets. First of all,

$$\begin{aligned} [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6 . \end{aligned} \tag{3.2}$$

Let $u = a_{12}a_{33} + a_{11}a_{34}$, $v = a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33}$. In view of (3.1) and (3.2) we have

$$(a_{22}y_5 - a_{12}y_6)\,u/\Delta^2 + (a_{11}y_6 - a_{21}y_5)\,v/\Delta^2 = y_6\,,$$

whence

$$\begin{cases} va_{11} - ua_{12} = \Delta^2\,, \\ va_{21} - ua_{22} = 0\,. \end{cases} \tag{3.3}$$

It follows that

$$\begin{cases} u = a_{21}\Delta\,, \\ v = a_{22}\Delta\,. \end{cases} \tag{3.4}$$

In addition,

$$\begin{aligned} [y_2, y_3] &= a_{21}a_{33}[x_1, x_3] + a_{22}a_{33}[x_2, x_3] + a_{21}a_{34}[x_1, x_4] + a_{22}a_{34}[x_2, x_4] \\ &= a_{21}a_{33}x_6 + a_{22}a_{33}(x_5 + tx_6) + a_{21}a_{34}x_5 + a_{22}a_{34}x_6 \\ &= (a_{22}a_{33} + a_{21}a_{34})x_5 + (a_{21}a_{33} + a_{22}a_{34} + ta_{22}a_{33})x_6 = y_5 + sy_6\,. \end{aligned} \tag{3.5}$$

Let $p = a_{22}a_{33} + a_{21}a_{34}$, $q = a_{21}a_{33} + a_{22}a_{34} + ta_{22}a_{33}$. In view of (3.1), (3.5)

$$(a_{22}y_5 - a_{12}y_6)\,p/\Delta^2 + (a_{11}y_6 - a_{21}y_5)\,q/\Delta^2 = y_5 + sy_6\,.$$

This implies that

$$\begin{cases} qa_{11} - pa_{12} = s\Delta^2\,, \\ qa_{21} - pa_{22} = -\Delta^2\,. \end{cases} \tag{3.6}$$

Consequently,

$$\begin{cases} p = (sa_{21} + a_{11})\,\Delta\,, \\ q = (sa_{22} + a_{12})\,\Delta\,. \end{cases} \tag{3.7}$$

Substituting u, v, p, q by the expressions given in (3.4), (3.7) we conclude that

$$\begin{cases} a_{11}a_{34} + a_{12}a_{33} = a_{21}\Delta\,, \\ a_{11}a_{33} + a_{12}(a_{34} + ta_{33}) = a_{22}\Delta\,, \\ a_{21}a_{34} + a_{22}a_{33} = (a_{11} + sa_{21})\,\Delta\,, \\ a_{21}a_{33} + a_{22}(a_{34} + ta_{33}) = (a_{12} + sa_{22})\,\Delta\,. \end{cases} \tag{3.8}$$

The first and the third equations of (3.8) yield

$$\begin{cases} a_{34} = a_{21}a_{22} - a_{12}(a_{11} + sa_{21})\,, \\ a_{33} = a_{11}(a_{11} + sa_{21}) - a_{21}a_{22}\,. \end{cases} \tag{3.9}$$

The two remaining ones yield

$$(3.10) \qquad \begin{cases} a_{33} = a_{22}a_{22} - a_{12}(a_{12} + sa_{22}), \\ a_{34} + ta_{33} = a_{11}(a_{12} + sa_{22}) - a_{21}a_{21}, \end{cases}$$

whence

$$(3.11) \qquad \begin{cases} a_{11}^2 + sa_{11}a_{21} - a_{21}^2 = a_{22}^2 - sa_{22}a_{12} - a_{12}^2 \neq 0, \\ 2(a_{11}a_{12} - a_{21}a_{22}) + s(a_{11}a_{22} - a_{21}a_{12}) = t(a_{11}^2 + sa_{11}a_{21} - a_{21}^2). \end{cases}$$

Let

$$\begin{cases} x_{11} = a_{11} + sa_{21}/2, \\ x_{12} = a_{12} + sa_{22}/2, \\ x_{21} = a_{21}, \\ x_{22} = a_{22}. \end{cases}$$

The system (3.11) can be rewritten in the form

$$(3.12) \qquad \begin{cases} x_{11}^2 - (1 + s^2/4)x_{21}^2 = -(x_{12}^2 - (1 + s^2/4)x_{22}^2) \neq 0, \\ 2(x_{11}x_{12} - (1 + s^2/4)x_{21}x_{22}) = t(x_{11}^2 - (1 + s^2/4)x_{21}^2). \end{cases}$$

Thus we may conclude that $\mathfrak{g}_t \cong \mathfrak{g}_s$ if and only if (3.12) has a rational solution such that $x_{11}x_{22} - x_{12}x_{21} \neq 0$. We state the following lemma in order to obtain less sophisticated conditions on s, t.

LEMMA 3.1. *Let $s, t \in \mathbf{Q}$. Then two conditions are equivalent:*
i) *there exists a matrix*

$$M = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathrm{GL}_2(\mathbf{Q})$$

such that x, y, z, w satisfy the system

$$(3.13) \qquad \begin{cases} x^2 - (1 + s^2/4)z^2 = -y^2 + (1 + s^2/4)w^2 \neq 0, \\ 2(xy - (1 + s^2/4)zw) = t(x^2 - (1 + s^2/4)z^2). \end{cases}$$

ii) *there exists $q \in \mathbf{Q}$ such that*

$$(3.14) \qquad (t^2 + 4)(s^2 + 4) = q^2.$$

Proof. Let $p = 1 + s^2/4$, $r = 1 + t^2/4$. The system (3.13) yields

$$(3.15) \qquad \begin{cases} x^2 + y^2 = p(z^2 + w^2), \\ 2xy - tx^2 = p(2zw - tz^2). \end{cases}$$

After the change of variables

$$x = x_0, \qquad y = \frac{1}{2}(y_0 + tx_0), \qquad z = \frac{z_0}{p}, \qquad w = \frac{1}{2}(w_0 + \frac{t}{p}z_0)$$

the system (3.15) can be rewritten as

$$\begin{cases} rx_0^2 + \frac{1}{4}y_0^2 = \frac{r}{p}z_0^2 + \frac{p}{4}w_0^2, \\ x_0y_0 = z_0w_0. \end{cases} \tag{3.16}$$

Geometrically, the system (3.16) defines the intersection I of two quadrics in the projective space $\mathbf{P}^3$. Let $\sigma\colon \mathbf{P}^1 \times \mathbf{P}^1 \to \mathbf{P}^3$ be the Segre map. In homogeneous coordinates $(a : b; \alpha : \beta)$ in $\mathbf{P}^1 \times \mathbf{P}^1$, σ is defined by $x_0 = a\alpha$, $y_0 = b\beta$, $z_0 = a\beta$, $w_0 = b\alpha$, and the image $\sigma(\mathbf{P}^1 \times \mathbf{P}^1)$ is the zero locus of the polynomial $x_0y_0 - z_0w_0$.

It is not hard to verify that in coordinates $(a : b;\, \alpha : \beta)$ the preimage $\sigma^{-1}(I)$ is given by the following equation (corresponding to the first one of (3.16)):

$$(4ra^2 - pb^2)(p\alpha^2 - \beta^2) = 0.$$

Thus $\sigma^{-1}(I)$ is the union of two pairs of lines (over $\mathbf{R}$). The second pair defined by the equation $p\alpha^2 - \beta^2 = 0$ yields $xw - zy = \det(M) = 0$. It follows that (3.15) has a rational solution if and only if the equation $4ra^2 - pb^2 = 0$ has one, i.e., $p/4r$ is the square of a rational number. This is equivalent to (3.14). Note that the condition $x^2 - pz^2 \neq 0$ in (3.13) is not very restrictive. This completes the proof of the lemma and of Theorem 1.

COROLLARY 3.2. *There are infinitely many non-isomorphic Lie algebras of the type* $\mathfrak{g}_s$ *over* $\mathbf{Q}$.

Proof. Let $s_1 = p_{11}$ be an odd prime. Consider $s_1^2 + 4 = p_1^2 + 4 = p_{21}^{n_{21}} \dots$ It is clear that $s_1^2 + 4$ is not a square (this means that at least one of the n_{2j} is odd) and is not divisible by p_{11}, whence all the $p_{2j} \neq p_{11}$. Let $s_2 = p_{11}p_{21} \dots$. It follows that

$$s_2^2 + 4 = p_{31}^{n_{31}} \dots$$

is not a square and is not divisible by p_{ij} where $i \leq 2$. Then we set $s_3 = p_{11}p_{21} \dots p_{31} \dots$ and so on.

In such a way we obtain an infinite sequence of numbers s_1, $s_2 \dots$ Let $i < j$. Note that $(s_i^2 + 4)(s_j^2 + 4) \neq q^2$, $q \in \mathbf{Q}$. Indeed, by the construction $(s_i^2 + 4)$ is divisible by some p and not divisible by p^2. Also, p divides s_j. Consequently, it does not divide $s_j^2 + 4$. This means that $(s_i^2 + 4)(s_j^2 + 4)$ is divisible by p but not by p^2, and this completes the proof.

4. Nilpotent Lie algebras with infinitely many non-isomorphic rational forms

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

4.1 Basic lemma

Let

$$\mathfrak{h} = \bigoplus_{i=1}^{c} \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over $\mathbf{Q}$ generated by $\mathfrak{h}_1$. Let $\mathbf{K}$ be a number field, $\dim_{\mathbf{Q}} \mathbf{K} = d$, of type (s,t), that is, there are s real and $2t$ complex embeddings of $\mathbf{K}$ in $\mathbf{C}$ $(d = s + 2t)$ whence there exists an isomorphism of $\mathbf{R}$-algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^{s} \mathbf{R} \oplus \bigoplus_{l=1}^{t} \mathbf{C}.$$

More generally one can take a finite-dimensional commutative associative algebra $\mathbf{A}$ over $\mathbf{Q}$ instead of $\mathbf{K}$. We consider the Lie algebra $\mathfrak{h}(\mathbf{K}) = \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}$ as a Lie algebra over $\mathbf{Q}$. This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^{s} \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^{t} \mathfrak{h}(\mathbf{C}),$$

i.e., $\mathfrak{h}(\mathbf{K})$ is a $\mathbf{Q}$-form of the last Lie algebra for any number field $\mathbf{K}$ of type (s,t). Secondly, there is an embedding $R\colon \mathbf{K}^* \to \mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$ of the multiplicative group $\mathbf{K}^*$ such that $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$ where $h_i \in \mathfrak{h}_i$ is homogenenous of degree i. The following lemma is straightforward.

Lemma 4.1. *Let $\mathbf{K} \neq \mathbf{K}'$ be two distinct number fields of the same type. If there is no injection of $\mathbf{K}^*$ into $\mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$ then two $\mathbf{Q}$-forms $\mathfrak{h}(\mathbf{K})$ and $\mathfrak{h}(\mathbf{K}')$ are not isomorphic.*

4.2 Proof of Theorem 2

We start with the class of nilpotence $c = 2$. Let $\mathbf{K} = \mathbf{Q}(\sqrt{m})$ and $\mathbf{K}' = \mathbf{Q}(\sqrt{n})$, where $m \neq n$ are two positive (resp. negative) square-free integers. Consider the automorphism $A = R(\sqrt{m})$ of $\mathfrak{h}(\mathbf{K}) = \mathfrak{f}_2(p,\mathbf{K})$. One immediately checks that

1) A^2 acts on $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}),\mathfrak{h}(\mathbf{K})]$ as $m \cdot Id$;

2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}),\mathfrak{h}(\mathbf{K})]} = m \cdot Id.$$

By Lemma 4.1 we must prove that there is no such automorphism for $\mathfrak{h}(\mathbf{K}') = \mathfrak{h}(\mathbf{Q}(\sqrt{n}))$. We choose the following basis of $\mathfrak{h}(\mathbf{K}')$ over $\mathbf{Q}$:

$$X_i = x_i \otimes 1, \quad Y_i = x_i \otimes \sqrt{n}, \quad C_{ij} = c_{ij} \otimes 1, \quad Z_{ij} = c_{ij} \otimes \sqrt{n},$$

$x_1, \dots, x_p, c_{ij} = [x_i, x_j]$ being the standard basis of $\mathfrak{f}_2(p, \mathbf{Q})$.

Suppose that there exists an automorphism A' with two above properties. First of all, let us show that $[X_i, A'(X_i)] = 0$. On the one hand,

$$A'[X_i, A'(X_i)] = [A'(X_i), mX_i] = -m[X_i, A'(X_i)].$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_i)].$$

Since the centralizer of X_i is generated modulo the centre by X_i, Y_i it follows that

$$A'(X_i) = p_i X_i + q_i Y_i + \varepsilon = x_i \otimes (p_i + q_i\sqrt{n}) + \varepsilon, \quad q_i \neq 0.$$

Here ε stands for a central element which plays no role below.

Consider now $[X_i, A'(X_j)] = c_{ij} \otimes (p_j + q_j\sqrt{n})$. On the one hand,

$$A'[X_i, A'(X_j)] = [A'(X_i), mX_j] = c_{ij} \otimes m(p_i + q_i\sqrt{n}).$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_j)] = c_{ij} \otimes m(p_j + q_j\sqrt{n}),$$

whence

$$p_i + q_i\sqrt{n} = p_j + q_j\sqrt{n} = p + q\sqrt{n} \notin \mathbf{Q} \;\; \forall\, i, j.$$

Finally, we apply A' to $[A'(X_i), A'(X_j)] = c_{ij} \otimes (p + q\sqrt{n})^2$. On the one hand,

$$A'[A'(X_i), A'(X_j)] = [mX_i, mX_j] = c_{ij} \otimes m^2.$$

On the other hand,

$$A'[A'(X_i), A'(X_i)] = m[A'(X_i), A'(X_j)] = c_{ij} \otimes m(p + q\sqrt{n})^2.$$

It follows that $m = (p + q\sqrt{n})^2$. We have obtained a contradiction since $q \neq 0$. Thus, there are infinitely many non-isomorphic rational forms of $\mathfrak{f}_2(p, \mathbf{R}) \oplus \mathfrak{f}_2(p, \mathbf{R})$ and of $\mathfrak{f}_2(p, \mathbf{C})$.

More generally let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 3$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (as a Lie algebra over $\mathbf{R}$) also have infinitely many non-isomorphic rational forms. Consider the automorphism A as above and note that it respects the descending central series. Any isomorphism between $\mathfrak{f}_c(p, \mathbf{K})$ and $\mathfrak{f}_c(p, \mathbf{K}')$ must respect it, too. Then we can take the free nilpotent quotients of class 2 of both algebras and obtain a contradiction just like in the first part of the proof. $\square$

Thus, the case of a free nilpotent Lie algebra $\mathfrak{f}_c(p, \mathbf{C})$ (as a Lie algebra over $\mathbf{R}$) on p generators differs from the case 2.2.

REMARK. All rational forms of $\mathfrak{f}_2(2, \mathbf{C}) = \mathfrak{hei}_3(\mathbf{C})$ and $\mathfrak{f}_2(2, \mathbf{R}) \oplus \mathfrak{f}_2(2, \mathbf{R}) = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ are listed in Theorem 3.

COROLLARY 4.2. *There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$ where $F_c(p, \mathbf{R})$ is the free nilpotent Lie group on p free generators.*

4.3 CLASSIFICATION OF RATIONAL FORMS FOR SOME 6-DIMENSIONAL LIE ALGEBRAS

Let m be a rational number and $A_m = \mathbf{Q}[x]/(x^2 - m)$. A_m is a 2-dimensional commutative algebra over $\mathbf{Q}$ which depends only on m modulo square factors. Thus there are four types of A_m :

1) if $m = 1$ then $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$;
2) if $m > 1$ is a positive square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is a real quadratic field over $\mathbf{Q}$;
3) if $m = 0$ then A_0 is the algebra of dual numbers over $\mathbf{Q}$;
4) if m is a negative square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is an imaginary quadratic field over $\mathbf{Q}$.

Let $\mathfrak{hei}_3(A_m)$ be a Heisenberg algebra over A_m considered over $\mathbf{Q}$. Then $\mathfrak{hei}_3(A_m)$ is a rational form of either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$. More precisely,

THEOREM 3. *Let $\mathfrak{h}$ be a 6-dimensional nilpotent Lie algebra of class 2 over $\mathbf{Q}$. Suppose that $[\mathfrak{h}, \mathfrak{h}]$ coincides with the 2-dimensional centre of $\mathfrak{h}$. Then $\mathfrak{h} \cong \mathfrak{hei}_3(A_m)$ for some $m \in \mathbf{Q}$ as above.*

Moreover,

1) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R}) = \mathfrak{g}_+$ *iff* $m > 0$,
2) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$ *iff* $m = 0$,
3) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{C}) = \mathfrak{g}_-$ *iff* $m < 0$,

and up to isomorphism there are no more rational forms for $\mathfrak{g}_-$, $\mathfrak{g}_0$, $\mathfrak{g}_+$. The Lie algebras $\mathfrak{hei}_3(A_m)$ and $\mathfrak{hei}_3(A_n)$ are isomorphic over $\mathbf{Q}$ if and only if A_m and A_n are isomorphic.

Proof. Take some $\mathbf{Q}$-basis $x_1, \ldots, x_6$ of $\mathfrak{h}$. First of all, we may suppose that $[x_1, x_2] = x_5$ (possibly after a change of basis). Thus x_5 is central. We have to deal with two cases.

CASE 1. All brackets $[x_1, x_j]$, $[x_2, x_j]$ $(j \geq 3)$ are multiples of x_5. If $[x_1, x_j] = a_j x_5$, $[x_2, x_j] = b_j x_5$ then we set $X_j = x_j - a_j x_2 + b_j x_1$ whence $[x_1, X_j] = [x_2, X_j] = 0$.

Since $[\mathfrak{h}, \mathfrak{h}]$ is 2-dimensional we conclude that some commutator, say $[x_3, x_4]$, is not a multiple of x_5 (for convenience, we use lower-case 'x' instead of 'X'). Consider

$$[x_3, x_4] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6 . \tag{4.1}$$

Commuting $[x_3, x_4]$ with x_1, x_2 we obtain that $a = b = 0$. Let us suppose that $f = 0$. Then

$$[x_3, x_4] = cx_3 + dx_4 + ex_5 . \tag{4.2}$$

Recall that x_5 and $[x_3, x_4]$ in the form (4.2) span the 2-dimensional centre. Commuting $cx_3 + dx_4 + ex_5$ from (4.2) with x_3, x_4 we get $c = d = 0$ and a contradiction. Thus $f \neq 0$. We may assume that $[x_3, x_4] = x_6$ where x_6 is central. Hence, we have the following multiplication table for $\mathfrak{h}$: $[x_1, x_2] = x_5$, $[x_3, x_4] = x_6$, other brackets being equal to 0. Consequently,

$$\mathfrak{h} = \langle x_1, x_2, x_5 \rangle \oplus \langle x_3, x_4, x_6 \rangle \cong \mathfrak{hei}_3(\mathbf{Q}) \oplus \mathfrak{hei}_3(\mathbf{Q}) .$$

CASE 2. Among the brackets $[x_1, x_j]$, $[x_2, x_j]$ $(j \geq 3)$ there is at least one which is not a multiple of x_5. In this case we may suppose (changing indices if necessary) that this bracket is $[x_1, x_3]$. Let

$$[x_1, x_3] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6 \quad . \tag{4.3}$$

and let us suppose that $d = f = 0$. Then

$$[x_1, x_3] = ax_1 + bx_2 + cx_3 + ex_5 . \tag{4.4}$$

Commuting the right-hand term of (4.4) with x_1 we get

$$0 = [x_1, [x_1, x_3]] = bx_5 + c[x_1, x_3] = cax_1 + cbx_2 + c^2 x_3 + (ce + b)x_5 .$$

Hence $c = b = 0$. By virtue of this $a = 0$ and we obtain a contradiction if we commute both sides of (4.4) with x_2. It follows that either $d \neq 0$ or $f \neq 0$. In other words, we may suppose that $[x_1, x_3]$ is equal to x_6.

Now

(4.5) $$[x_1, x_2] = x_5, \quad [x_1, x_3] = x_6$$

where x_5, x_6 span $[\mathfrak{h}, \mathfrak{h}]$. Suppose that $[x_2, x_3] = ax_5 + bx_6$. Adding if necessary some multiples of x_1 to x_2 and x_3 we obtain $[x_2, x_3] = 0$. In the same way we may suppose that $[x_1, x_4] = 0$. Adding to x_4 some multiple of x_1 we also obtain a relation $[x_2, x_4] = Cx_6$. Moreover, after scaling x_4 we get $C = 0$ or $C = 1$. Thus, $\mathfrak{h}$ has a basis in which the *non-trivial* brackets are the following:

(4.6) $$\begin{aligned} &[x_1, x_2] = x_5, && [x_1, x_3] = x_6, \\ &[x_2, x_4] = Cx_6 \quad (C = 0 \text{ or } C = 1), && [x_3, x_4] = Ax_5 + Bx_6. \end{aligned}$$

In any case $A^2 + B^2 + C^2 \neq 0$ because x_4 cannot belong to the 2-dimensional centre of $\mathfrak{h}$.

We will show that we can always make $C = 1$ and $B = 0$ in (4.6).

SUBCASE 2.1. If $C = 0$ then the following basis transformation

(4.7) $$\begin{aligned} &X_1 = x_1, && X_2 = ax_2 + x_3, \\ &X_3 = Ax_2 + Bx_3, && X_4 = x_4, \end{aligned}$$

yields (a is any constant such that $aB \neq A$)

(4.8) $$\begin{aligned} &[X_1, X_2] = ax_5 + x_6 = X_5, && [X_1, X_3] = Ax_5 + Bx_6 = X_6, \\ &[X_2, X_4] = Ax_5 + Bx_6 = X_6, && [X_3, X_4] = B(Ax_5 + Bx_6) = BX_6. \end{aligned}$$

From now on we may suppose that $C = 1$ in (4.6) and we arrive at

SUBCASE 2.2: $C = 1$, $A = 0$. Let

(4.9) $$\begin{aligned} &X_1 = x_1 + ax_4, && X_2 = x_2 - ax_3, \\ &X_3 = x_2 + dx_3, && X_4 = -x_1 + dx_4, \end{aligned}$$

where $a, d, a + d \neq 0$, $aB \neq 1$, $dB \neq -1$. Hence

(4.10) $$\begin{aligned} &[X_1, X_2] = x_5 + (a^2B - 2a)x_6 = X_5, \\ &[X_1, X_3] = x_5 + (d - a - adB)x_6 = X_6, \\ &[X_2, X_4] = x_5 + (d - a - adB)x_6 = X_6, \\ &[X_3, X_4] = x_5 + (d^2B + 2d)x_6 = \lambda X_5 + (1 - \lambda)X_6. \end{aligned}$$

Since a, d and $a + d$ are all non-zero, X_5 and X_6 are linearly independent. Straightforward computations yield

$$\lambda = \frac{dB + 1}{aB - 1} \neq 0, 1.$$

Thus we have the following alternative.

SUBCASE 2.3.1: $C = 1$; A, B, $4A + B^2 \neq 0$. Let now

$$X_1 = x_1 + tx_4\,, \qquad X_2 = x_2 - tx_3\,,$$
$$X_3 = x_3\,, \qquad X_4 = x_4\,, \tag{4.11}$$

where $t = -B/2A$. Hence

$$\begin{aligned} [X_1, X_2] &= (1 + t^2 A)x_5 + (t^2 B - 2t)x_6 = X_5\,, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6\,, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6\,, \\ [X_3, X_4] &= Ax_5 + Bx_6 = \alpha X_5 = \tfrac{4A^2}{4A+B^2} X_5\,. \end{aligned} \tag{4.12}$$

SUBCASE 2.3.2: $C = 1$; $A, B \neq 0$, $4A + B^2 = 0$. The same transformation (4.11) yields

$$\begin{aligned} [X_1, X_2] &= 0\,, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6\,, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6\,, \\ [X_3, X_4] &= Ax_5 + Bx_6 = X_5 \end{aligned} \tag{4.13}$$

and, after the transformation $x_1 = X_3$, $x_2 = X_4$, $x_3 = X_1$, $x_4 = X_2$, $x_5 = X_5$, $x_6 = -X_6$, we obtain (4.12) with $\alpha = 0$. Anyway, we obtain the desired form of $\mathfrak{h}$

$$[x_1, x_2] = x_5\,, \quad [x_1, x_3] = x_6\,, \quad [x_2, x_4] = x_6\,, \quad [x_3, x_4] = Ax_5\,. \tag{4.14}$$

Scaling x_3, x_4 by $\lambda \neq 0$ we may suppose that $A = m$ where m is a square-free integer as above.

In order to conclude the proof of the first part of the theorem we point out an isomorphism $\rho\colon \mathfrak{h} \to \mathfrak{hei}_3(A_m)$. Recall that A_m has a basis 1, x over $\mathbf{Q}$ such that $x^2 = m$. Here are the matrices representing $\rho(x_i)$ if $m \neq 1$ (the case $m = 1$ is left to the reader as an easy exercise):

$$\begin{aligned} &\rho(x_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &\rho(x_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_4) = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_6) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.15}$$

Now it is evident that $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$ depending on the sign of m. Thus, we have classified up to $\mathbf{Q}$-isomorphism all rational forms for these 3 real Lie algebras. By Theorem 2 these forms are non-isomorphic. The proof of the theorem is complete. □

REMARK. It is worth mentioning that the above three real Lie algebras are not pairwise isomorphic over $\mathbf{R}$. Indeed, the centralizer of any element in $\mathfrak{g}_- = \mathfrak{hei}_3(\mathbf{C})$ is even dimensional over $\mathbf{R}$ since this algebra can be viewed as a complex Lie algebra, whereas in both $\mathfrak{g}_+ = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ and $\mathfrak{g}_0 = \mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$ there are elements with 5-dimensional centralizers. In order to show that the last two algebras are not isomorphic we need some more information about elements with 5-dimensional centralizers.

The centralizer $C(x)$ will not be changed if we scale x by any $\lambda \neq 0$ or add to x any central element. This means that dimension of the centralizer is a well-defined function on the projective space $\mathbf{P}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ where $\mathfrak{g}$ is either $\mathfrak{g}_+$ or $\mathfrak{g}_0$. Straightforward computations show that in $\mathbf{P}(\mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0])$ all points with 5-dimensional centralizer belong to a unique line whereas in $\mathbf{P}(\mathfrak{g}_+/[\mathfrak{g}_+,\mathfrak{g}_+])$ the points under consideration form two disjoint lines.

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