2. Nilpotent Lie algebras with a unique rational form up to isomorphism

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THEOREM 2. *Let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 2$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (regarded over $\mathbf{R}$) have infinitely many non-isomorphic rational forms.*

In Theorem 3 we also classify all rational forms for three 6-dimensional real nilpotent Lie algebras $\mathfrak{g}$ (two of them appear in Theorem 2 for $p = c = 2$) which are of class 2 and have 2-dimensional centre coinciding with the derived subalgebra.

In conclusion let us mention a direct way to prove that two given lattices in a nilpotent Lie group are not commensurable. For example, let $G = UT_3(\mathbf{R})$ be the Lie group of upper triangular 3×3-matrices with 1 on the diagonal, $\mathfrak{g} = \mathfrak{f}_2(2, \mathbf{R})$ be Lie algebra of G. Consider $G \times G$ and its Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ which has infinitely many non-isomorphic rational forms $\mathfrak{h}_m$ ($m \geq 1$ is a square-free integer), in view and in the notation of Theorems 2, 3 (see Section 4 for more details).

Let Γ_m and Γ_n be corresponding lattices in $G \times G$ for distinct m, n. One can prove that the ratio of the covolumes of Γ_m and Γ_n with respect to a Haar measure on $G \times G$ equals $m\sqrt{m}/n\sqrt{n}$ up to a rational factor. Hence the lattices are not commensurable. Note that by Proposition 1.1 and Theorem 3 Γ_m and Γ_n are not commensurable in any sense.

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2. NILPOTENT LIE ALGEBRAS WITH A UNIQUE RATIONAL FORM UP TO ISOMORPHISM

2.1 HEISENBERG ALGEBRAS

Let us begin with the following considerations that we will use here and in the next sections (see [2, Chapter 5] for more details). Suppose that a real Lie algebra $\mathfrak{g}$ has a $\mathbf{Q}$-form $\mathfrak{h}$ and $\mathfrak{i}$ (resp. $\mathfrak{a}$) is an ideal (resp. a subalgebra) of $\mathfrak{g}$. We say that $\mathfrak{i}$ (resp. $\mathfrak{a}$) is rational if $\mathfrak{i} \cap \mathfrak{h}$ (resp. $\mathfrak{a} \cap \mathfrak{h}$) is a rational form of $\mathfrak{i}$ (resp. $\mathfrak{a}$). For instance, the terms $C^k\mathfrak{g}$ of the lower central series of $\mathfrak{g}$ are rational as well as centralizers of rational subalgebras or ideals. It is not hard to see that $\mathfrak{h}/\mathfrak{i} \cap \mathfrak{h}$ is a rational form of the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$.

Let

$$\mathfrak{g} = \mathfrak{i}_1 > \mathfrak{i}_2 > \cdots > \mathfrak{i}_{k+1} = 0 \tag{2.1}$$

be a descending series of rational ideals of $\mathfrak{g}$. We say that a basis $X = \{x_1, \dots, x_d\}$ of a rational form $\mathfrak{h}$ is based on (2.1) if $x_1, \dots, x_{p_1}$ generate $\mathfrak{g} \bmod \mathfrak{i}_2$, $x_1, \dots, x_{p_2}$ generate $\mathfrak{g} \bmod \mathfrak{i}_3$ and so on. It can be shown that such a basis exists for any series (2.1). In the sequel we will use these kinds of bases for a suitable descending series dealing, for instance, with Heisenberg algebras.

Recall that the (generalized) Heisenberg algebra $\mathfrak{hei}_{2k+1}(\mathbf{R})$ has an $\mathbf{R}$-basis $H_1, \dots, H_{2k+1}$ in which

$$[H_1, H_2] = [H_3, H_4] = \cdots = [H_{2k-1}, H_{2k}] = H_{2k+1}, \tag{2.2}$$

other brackets being trivial. Here the 1-dimensional centre is spanned by H_{2k+1}.

Given an extension (1.1) one can attach to it a 2-cocycle $\omega\colon \Lambda^2\mathfrak{a} \to \mathbf{R}$ in the usual way. Also ω can be regarded as a symplectic form on $\mathfrak{a}$. If $\mathfrak{b} = \mathfrak{hei}_{2k+1}(\mathbf{R})$ then ω is the canonical non-degenerate symplectic form with respect to the basis $H_1, \dots, H_{2k}$ (mod $\mathbf{R} \cdot H_{2k+1}$).

Let $d = \dim_{\mathbf{R}} \mathfrak{a}$ and let $m = d - rank(\omega)$ be the codimension of the kernel of ω. It is not hard to see (cf. the proof of the proposition below) that the Lie algebra $\mathfrak{b}$ is uniquely defined up to $\mathbf{R}$-isomorphism by d and m. Namely,

$$\mathfrak{b} \cong \mathfrak{hei}_{d+1-m}(\mathbf{R}) \oplus \mathbf{R}^m.$$

This implies that the centre of $\mathfrak{b}$ is $(m+1)$-dimensional. Thus, two Lie algebras $\mathfrak{b}_1$ and $\mathfrak{b}_2$ ($\dim_{\mathbf{R}} \mathfrak{b}_1 = \dim_{\mathbf{R}} \mathfrak{b}_2$) of type (1.1) are not isomorphic if $m_1 \neq m_2$.

Evidently, $\mathfrak{hei}_{2k+1}(\mathbf{Q})$ is a rational form for $\mathfrak{hei}_{2k+1}(\mathbf{R})$.

The following proposition holds.

PROPOSITION 2.1. *In the above notation let* $\mathfrak{h}$ *be a rational form of* $\mathfrak{b}$. *Let* $d = \dim(\mathfrak{b}) - 1$, $m = \dim[\mathfrak{b}, \mathfrak{b}]$ *and let* $\mathbf{Q}^m$ *denote the abelian Lie* $\mathbf{Q}$*-algebra of dimension* m. *Then*

$$\mathfrak{h} \cong \mathfrak{hei}_{d+1-m}(\mathbf{Q}) \oplus \mathbf{Q}^m$$

over $\mathbf{Q}$, *i.e., there is a unique rational form for* $\mathfrak{b}$ *up to isomorphism.*

Proof. Choose a $\mathbf{Q}$-basis $B_1, \dots, B_{d+1}$ for $\mathfrak{h}$. Either all brackets $[B_i, B_j] = 0$, and then $\mathfrak{h} \cong \mathbf{Q}^{d+1}$, or there are i, j such that $[B_i, B_j] = C \neq 0$.

We may suppose that $C = B_{d+1}$. Thus the derived subalgebra of $\mathfrak{h}$ is spanned by B_{d+1}. The corresponding symplectic form ω is represented by a skew-symmetric $d \times d$ matrix $M = (\mu_{ij})$ with respect to the basis $B_1, \dots, B_d$ (mod $[\mathfrak{h}, \mathfrak{h}]$). Namely, $[B_i, B_j] = \mu_{ij} B_{d+1}$. Over $\mathbf{Q}$ one can choose a canonical symplectic basis $\widehat{B_1}, \dots, \widehat{B_d}$ (mod $[\mathfrak{h}, \mathfrak{h}]$) so that the matrix $\widehat{M}$ representing ω has l blocks of type

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

standing on the diagonal, the other entries being trivial. The rank of ω is equal to $2l$ and $2l = d - m$. In the basis $B_1, \dots, B_{d+1}$ (we omit the 'hats') of $\mathfrak{h}$

$$[B_1, B_2] = [B_3, B_4] = \cdots = [B_{2l-1}, B_{2l}] = B_{d+1},$$

all the other brackets being trivial. This completes the proof.

2.2 Example of a free nilpotent algebra

Let $\mathfrak{f}_c(n, \mathbf{R})$ be the free nilpotent Lie algebra of class c on n generators. Then $\mathfrak{f}_c(n, \mathbf{R})$ has a unique rational form $\mathfrak{f}_c(n, \mathbf{Q})$ up to isomorphism (cf. Theorem 2).

Indeed, let $\mathfrak{h} = \langle x_1, \dots, x_n, \dots \rangle$ be a rational form of $\mathfrak{f}_c(n, \mathbf{R})$. We may suppose that $x_1, \dots, x_n$ span (modulo the derived subalgebra) $\mathfrak{h}/[\mathfrak{h}, \mathfrak{h}] \cong \mathbf{Q}^n$. Consequently, $\mathfrak{h}$ is generated by $\{x_1, \dots, x_n\}$ as a Lie algebra. There exists an epimorphism $\pi \colon \mathfrak{f}_c(n, \mathbf{Q}) \to \mathfrak{h}$ because $\mathfrak{f}_c(n, \mathbf{Q})$ is free. It must be an isomorphism since the dimension of $\mathfrak{h}$ equals the dimension (not depending on the ground field) of a free nilpotent Lie algebra of class c on n generators.

2.3 More examples

The purpose of this subsection is to sketch two more examples of Lie algebras with a unique rational form up to isomorphism.

Let $\mathfrak{g}_t$, $t \in \mathbf{R}$, be a family of real 6-dimensional Lie algebras with a basis $\{x_1, \dots, x_6\}$ such that

$$[x_1, x_2] = x_3, \quad [x_1, x_3] = t x_5, \quad [x_1, x_5] = x_6,$$
$$[x_2, x_3] = x_4, \quad [x_2, x_4] = x_5, \quad [x_3, x_4] = x_6,$$

other brackets being trivial. One can show that

1. $C^k \mathfrak{g}_t = \langle x_{k+1}, \dots, x_6 \rangle$, $k = 2, \dots, 5$, where $C^k \mathfrak{g}_t$ are the terms of the lower central series of $\mathfrak{g}_t$.
2. The centralizer $\mathfrak{C}$ of $C^4 \mathfrak{g}_t$, that is, $\mathfrak{C} = \{c \in \mathfrak{g}_t \mid [c, C^4 \mathfrak{g}_t] = 0\}$ is spanned by $x_2, \dots, x_6$.

3. Real Lie algebras $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are not isomorphic but $\forall t \neq 0 \quad \mathfrak{g}_t \cong \mathfrak{g}_1$.
4. If $t \in \mathbf{Q} \setminus \{0\}$ then the rational algebra $\mathfrak{g}_t \cong \mathfrak{g}_1$ over $\mathbf{Q}$.
5. $\mathfrak{g}_0$ and $\mathfrak{g}_1$ are two Lie algebras with a unique rational form up to isomorphism.
6. Let $\mathfrak{g}$ be a split real simple Lie algebra of type G_2, $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then $\mathfrak{n}_+$ is isomorphic to $\mathfrak{g}_0$.

3. Malcev's example

In this Section we develop Malcev's example and prove Theorem 1.

Suppose that there is a $\mathbf{Q}$-isomorphism between $\mathfrak{g}_t$ and $\mathfrak{g}_s$. It must be written in the following form (cf. [5]) since $C^2\mathfrak{g}_t = \langle x_4, x_5, x_6 \rangle$, $C^3\mathfrak{g}_t = \langle x_5, x_6 \rangle$ and the centralizer $\mathfrak{c}$ of $C^2\mathfrak{g}$, which is an ideal in this case, is spanned by $x_3, \dots, x_6$.

$$\begin{cases} y_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 + \dots \\ y_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4 + \dots \\ y_3 = a_{33}x_3 + a_{34}x_4 + \dots \\ y_4 = a_{44}x_4 + \dots \end{cases}$$

We do not explicit the expressions for y_5, y_6. Here $y_1, \dots, y_6$ are basis elements of $\mathfrak{g}_s$ satisfying the relations (1.2).

We obtain after straightforward computations that

$$[y_1, y_2] = y_4 = \Delta x_4 + \dots ,$$

$\Delta = a_{11}a_{22} - a_{12}a_{21} = a_{44} \neq 0$. On the other hand,

$$\begin{cases} y_5 = [y_1, y_4] = \Delta(a_{11}x_5 + a_{12}x_6), \\ y_6 = [y_2, y_4] = \Delta(a_{21}x_5 + a_{22}x_6). \end{cases}$$

Hence,

$$\begin{cases} x_5 = (a_{22}y_5 - a_{12}y_6)/\Delta^2, \\ x_6 = (a_{11}y_6 - a_{21}y_5)/\Delta^2. \end{cases} \tag{3.1}$$

We need to compute the remaining two brackets. First of all,

$$\begin{aligned} [y_1, y_3] &= a_{11}a_{33}[x_1, x_3] + a_{12}a_{33}[x_2, x_3] + a_{11}a_{34}[x_1, x_4] + a_{12}a_{34}[x_2, x_4] \\ &= a_{11}a_{33}x_6 + a_{12}a_{33}(x_5 + tx_6) + a_{11}a_{34}x_5 + a_{12}a_{34}x_6 \\ &= (a_{12}a_{33} + a_{11}a_{34})x_5 + (a_{11}a_{33} + a_{12}a_{34} + ta_{12}a_{33})x_6 = y_6. \end{aligned} \tag{3.2}$$