4. Nilpotent Lie algebras with infinitely many non-isomorphic rational forms

Objekttyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.08.2024**

4. NILPOTENT LIE ALGEBRAS WITH INFINITELY MANY NON-ISOMORPHIC RATIONAL FORMS

In this section we propose a construction which can provide a series of nilpotent Lie algebras with infinitely many isomorphism classes of rational forms.

4.1 BASIC LEMMA

Let

$$\mathfrak{h} = \bigoplus_{i=1}^{c} \mathfrak{h}_i = \mathfrak{h}(\mathbf{Q})$$

be a graded Lie algebra over $\mathbf{Q}$ generated by $\mathfrak{h}_1$. Let $\mathbf{K}$ be a number field, $\dim_{\mathbf{Q}} \mathbf{K} = d$, of type (s,t), that is, there are s real and $2t$ complex embeddings of $\mathbf{K}$ in $\mathbf{C}$ $(d = s + 2t)$ whence there exists an isomorphism of $\mathbf{R}$-algebras

$$\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R} \cong \bigoplus_{k=1}^{s} \mathbf{R} \oplus \bigoplus_{l=1}^{t} \mathbf{C}\,.$$

More generally one can take a finite-dimensional commutative associative algebra $\mathbf{A}$ over $\mathbf{Q}$ instead of $\mathbf{K}$. We consider the Lie algebra $\mathfrak{h}(\mathbf{K}) = \mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}$ as a Lie algebra over $\mathbf{Q}$. This algebra has two important properties. Firstly,

$$\mathfrak{h}(\mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong (\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{K}) \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{h} \otimes_{\mathbf{Q}} (\mathbf{K} \otimes_{\mathbf{Q}} \mathbf{R}) \cong \bigoplus_{k=1}^{s} \mathfrak{h}(\mathbf{R}) \oplus \bigoplus_{l=1}^{t} \mathfrak{h}(\mathbf{C})\,,$$

i.e., $\mathfrak{h}(\mathbf{K})$ is a $\mathbf{Q}$-form of the last Lie algebra for any number field $\mathbf{K}$ of type (s,t). Secondly, there is an embedding $R \colon \mathbf{K}^* \to \mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}))$ of the multiplicative group $\mathbf{K}^*$ such that $R(k)(h_i \otimes k_1) = h_i \otimes \tilde{k}k^i$ where $h_i \in \mathfrak{h}_i$ is homogenenous of degree i. The following lemma is straightforward.

LEMMA 4.1. *Let $\mathbf{K} \neq \mathbf{K}'$ be two distinct number fields of the same type. If there is no injection of $\mathbf{K}^*$ into $\mathrm{Aut}_{\mathbf{Q}}(\mathfrak{h}(\mathbf{K}'))$ then two $\mathbf{Q}$-forms $\mathfrak{h}(\mathbf{K})$ and $\mathfrak{h}(\mathbf{K}')$ are not isomorphic.*

4.2 PROOF OF THEOREM 2

We start with the class of nilpotence $c = 2$. Let $\mathbf{K} = \mathbf{Q}(\sqrt{m})$ and $\mathbf{K}' = \mathbf{Q}(\sqrt{n})$, where $m \neq n$ are two positive (resp. negative) square-free integers. Consider the automorphism $A = R(\sqrt{m})$ of $\mathfrak{h}(\mathbf{K}) = \mathfrak{f}_2(p, \mathbf{K})$. One immediately checks that

1) A^2 acts on $\mathfrak{h}(\mathbf{K})/[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]$ as $m \cdot Id$;

2) the restriction

$$A|_{[\mathfrak{h}(\mathbf{K}), \mathfrak{h}(\mathbf{K})]} = m \cdot Id\,.$$

By Lemma 4.1 we must prove that there is no such automorphism for $\mathfrak{h}(\mathbf{K}') = \mathfrak{h}(\mathbf{Q}(\sqrt{n}))$. We choose the following basis of $\mathfrak{h}(\mathbf{K}')$ over $\mathbf{Q}$:

$$X_i = x_i \otimes 1, \quad Y_i = x_i \otimes \sqrt{n}, \quad C_{ij} = c_{ij} \otimes 1, \quad Z_{ij} = c_{ij} \otimes \sqrt{n},$$

$x_1, \dots, x_p, c_{ij} = [x_i, x_j]$ being the standard basis of $\mathfrak{f}_2(p, \mathbf{Q})$.

Suppose that there exists an automorphism A' with two above properties. First of all, let us show that $[X_i, A'(X_i)] = 0$. On the one hand,

$$A'[X_i, A'(X_i)] = [A'(X_i), mX_i] = -m[X_i, A'(X_i)].$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_i)].$$

Since the centralizer of X_i is generated modulo the centre by X_i, Y_i it follows that

$$A'(X_i) = p_i X_i + q_i Y_i + \varepsilon = x_i \otimes (p_i + q_i\sqrt{n}) + \varepsilon, \quad q_i \neq 0.$$

Here ε stands for a central element which plays no role below.

Consider now $[X_i, A'(X_j)] = c_{ij} \otimes (p_j + q_j\sqrt{n})$. On the one hand,

$$A'[X_i, A'(X_j)] = [A'(X_i), mX_j] = c_{ij} \otimes m(p_i + q_i\sqrt{n}).$$

On the other hand,

$$A'[X_i, A'(X_i)] = m[X_i, A'(X_j)] = c_{ij} \otimes m(p_j + q_j\sqrt{n}),$$

whence

$$p_i + q_i\sqrt{n} = p_j + q_j\sqrt{n} = p + q\sqrt{n} \notin \mathbf{Q} \ \forall i, j.$$

Finally, we apply A' to $[A'(X_i), A'(X_j)] = c_{ij} \otimes (p + q\sqrt{n})^2$. On the one hand,

$$A'[A'(X_i), A'(X_j)] = [mX_i, mX_j] = c_{ij} \otimes m^2.$$

On the other hand,

$$A'[A'(X_i), A'(X_i)] = m[A'(X_i), A'(X_j)] = c_{ij} \otimes m(p + q\sqrt{n})^2.$$

It follows that $m = (p + q\sqrt{n})^2$. We have obtained a contradiction since $q \neq 0$. Thus, there are infinitely many non-isomorphic rational forms of $\mathfrak{f}_2(p, \mathbf{R}) \oplus \mathfrak{f}_2(p, \mathbf{R})$ and of $\mathfrak{f}_2(p, \mathbf{C})$.

More generally let $\mathfrak{g} = \mathfrak{f}_c(p, \mathbf{R})$ be a free nilpotent Lie algebra of class $c \geq 3$ on p generators. Then $\mathfrak{g} \oplus \mathfrak{g}$ and $\mathfrak{g} \otimes_{\mathbf{R}} \mathbf{C} = \mathfrak{f}_c(p, \mathbf{C})$ (as a Lie algebra over $\mathbf{R}$) also have infinitely many non-isomorphic rational forms. Consider the automorphism A as above and note that it respects the descending central series. Any isomorphism between $\mathfrak{f}_c(p, \mathbf{K})$ and $\mathfrak{f}_c(p, \mathbf{K}')$ must respect it, too. Then we can take the free nilpotent quotients of class 2 of both algebras and obtain a contradiction just like in the first part of the proof. □

Thus, the case of a free nilpotent Lie algebra $\mathfrak{f}_c(p, \mathbf{C})$ (as a Lie algebra over $\mathbf{R}$) on p generators differs from the case 2.2.

REMARK. All rational forms of $\mathfrak{f}_2(2, \mathbf{C}) = \mathfrak{hei}_3(\mathbf{C})$ and $\mathfrak{f}_2(2, \mathbf{R}) \oplus \mathfrak{f}_2(2, \mathbf{R}) = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ are listed in Theorem 3.

COROLLARY 4.2. *There are infinitely many non-commensurable (in any sense) lattices in the Lie groups of type* $F_c(p, \mathbf{R}) \times F_c(p, \mathbf{R})$ *where* $F_c(p, \mathbf{R})$ *is the free nilpotent Lie group on p free generators.*

4.3 CLASSIFICATION OF RATIONAL FORMS FOR SOME 6-DIMENSIONAL LIE ALGEBRAS

Let m be a rational number and $A_m = \mathbf{Q}[x]/(x^2 - m)$. A_m is a 2-dimensional commutative algebra over $\mathbf{Q}$ which depends only on m modulo square factors. Thus there are four types of A_m :

1) if $m = 1$ then $A_m \cong \mathbf{Q} \oplus \mathbf{Q}$;
2) if $m > 1$ is a positive square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is a real quadratic field over $\mathbf{Q}$;
3) if $m = 0$ then A_0 is the algebra of dual numbers over $\mathbf{Q}$;
4) if m is a negative square-free integer then $A_m \cong \mathbf{Q}(\sqrt{m})$ is an imaginary quadratic field over $\mathbf{Q}$.

Let $\mathfrak{hei}_3(A_m)$ be a Heisenberg algebra over A_m considered over $\mathbf{Q}$. Then $\mathfrak{hei}_3(A_m)$ is a rational form of either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$. More precisely,

THEOREM 3. *Let* $\mathfrak{h}$ *be a 6-dimensional nilpotent Lie algebra of class 2 over* $\mathbf{Q}$. *Suppose that* $[\mathfrak{h}, \mathfrak{h}]$ *coincides with the 2-dimensional centre of* $\mathfrak{h}$. *Then* $\mathfrak{h} \cong \mathfrak{hei}_3(A_m)$ *for some* $m \in \mathbf{Q}$ *as above.*

Moreover,

1) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R}) = \mathfrak{g}_+$ *iff* $m > 0$,
2) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{R}[x]/(x^2)) = \mathfrak{g}_0$ *iff* $m = 0$,
3) $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R} \cong \mathfrak{hei}_3(\mathbf{C}) = \mathfrak{g}_-$ *iff* $m < 0$,

and up to isomorphism there are no more rational forms for $\mathfrak{g}_-$, $\mathfrak{g}_0$, $\mathfrak{g}_+$. *The Lie algebras* $\mathfrak{hei}_3(A_m)$ *and* $\mathfrak{hei}_3(A_n)$ *are isomorphic over* $\mathbf{Q}$ *if and only if* A_m *and* A_n *are isomorphic.*

Proof. Take some $\mathbf{Q}$-basis $x_1, \dots, x_6$ of $\mathfrak{h}$. First of all, we may suppose that $[x_1, x_2] = x_5$ (possibly after a change of basis). Thus x_5 is central. We have to deal with two cases.

CASE 1. All brackets $[x_1, x_j]$, $[x_2, x_j]$ ($j \geq 3$) are multiples of x_5. If $[x_1, x_j] = a_j x_5$, $[x_2, x_j] = b_j x_5$ then we set $X_j = x_j - a_j x_2 + b_j x_1$ whence $[x_1, X_j] = [x_2, X_j] = 0$.

Since $[\mathfrak{h}, \mathfrak{h}]$ is 2-dimensional we conclude that some commutator, say $[x_3, x_4]$, is not a multiple of x_5 (for convenience, we use lower-case 'x' instead of 'X'). Consider

$$[x_3, x_4] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6 \,. \tag{4.1}$$

Commuting $[x_3, x_4]$ with x_1, x_2 we obtain that $a = b = 0$. Let us suppose that $f = 0$. Then

$$[x_3, x_4] = cx_3 + dx_4 + ex_5 \,. \tag{4.2}$$

Recall that x_5 and $[x_3, x_4]$ in the form (4.2) span the 2-dimensional centre. Commuting $cx_3 + dx_4 + ex_5$ from (4.2) with x_3, x_4 we get $c = d = 0$ and a contradiction. Thus $f \neq 0$. We may assume that $[x_3, x_4] = x_6$ where x_6 is central. Hence, we have the following multiplication table for $\mathfrak{h}$: $[x_1, x_2] = x_5$, $[x_3, x_4] = x_6$, other brackets being equal to 0. Consequently,

$$\mathfrak{h} = \langle x_1, x_2, x_5 \rangle \oplus \langle x_3, x_4, x_6 \rangle \cong \mathfrak{hei}_3(\mathbf{Q}) \oplus \mathfrak{hei}_3(\mathbf{Q}) \,.$$

CASE 2. Among the brackets $[x_1, x_j]$, $[x_2, x_j]$ ($j \geq 3$) there is at least one which is not a multiple of x_5. In this case we may suppose (changing indices if necessary) that this bracket is $[x_1, x_3]$. Let

$$[x_1, x_3] = ax_1 + bx_2 + cx_3 + dx_4 + ex_5 + fx_6 \quad . \tag{4.3}$$

and let us suppose that $d = f = 0$. Then

$$[x_1, x_3] = ax_1 + bx_2 + cx_3 + ex_5 \,. \tag{4.4}$$

Commuting the right-hand term of (4.4) with x_1 we get

$$0 = [x_1, [x_1, x_3]] = bx_5 + c[x_1, x_3] = cax_1 + cbx_2 + c^2 x_3 + (ce + b)x_5 \,.$$

Hence $c = b = 0$. By virtue of this $a = 0$ and we obtain a contradiction if we commute both sides of (4.4) with x_2. It follows that either $d \neq 0$ or $f \neq 0$. In other words, we may suppose that $[x_1, x_3]$ is equal to x_6.

Now

$$[x_1,x_2]=x_5,\quad [x_1,x_3]=x_6 \tag{4.5}$$

where x_5, x_6 span $[\mathfrak{h},\mathfrak{h}]$. Suppose that $[x_2,x_3] = ax_5+bx_6$. Adding if necessary some multiples of x_1 to x_2 and x_3 we obtain $[x_2,x_3]=0$. In the same way we may suppose that $[x_1,x_4]=0$. Adding to x_4 some multiple of x_1 we also obtain a relation $[x_2,x_4] = Cx_6$. Moreover, after scaling x_4 we get $C=0$ or $C=1$. Thus, $\mathfrak{h}$ has a basis in which the *non-trivial* brackets are the following:

$$\begin{aligned} &[x_1,x_2]=x_5\,, && [x_1,x_3]=x_6\,,\\ &[x_2,x_4]=Cx_6 \quad (C=0 \text{ or } C=1)\,, && [x_3,x_4]=Ax_5+Bx_6\,. \end{aligned} \tag{4.6}$$

In any case $A^2+B^2+C^2\neq 0$ because x_4 cannot belong to the 2-dimensional centre of $\mathfrak{h}$.

We will show that we can always make $C=1$ and $B=0$ in (4.6).

SUBCASE 2.1. If $C=0$ then the following basis transformation

$$\begin{aligned} &X_1=x_1\,, && X_2=ax_2+x_3\,,\\ &X_3=Ax_2+Bx_3\,, && X_4=x_4\,, \end{aligned} \tag{4.7}$$

yields (a is any constant such that $aB\neq A$)

$$\begin{aligned} &[X_1,X_2]=ax_5+x_6=X_5\,, && [X_1,X_3]=Ax_5+Bx_6=X_6\,,\\ &[X_2,X_4]=Ax_5+Bx_6=X_6\,, && [X_3,X_4]=B(Ax_5+Bx_6)=BX_6\,. \end{aligned} \tag{4.8}$$

From now on we may suppose that $C=1$ in (4.6) and we arrive at

SUBCASE 2.2: $C=1,\ A=0.$ Let

$$\begin{aligned} &X_1=x_1+ax_4\,, && X_2=x_2-ax_3\,,\\ &X_3=x_2+dx_3\,, && X_4=-x_1+dx_4\,, \end{aligned} \tag{4.9}$$

where $a,d,a+d\neq 0$, $aB\neq 1$, $dB\neq -1$. Hence

$$\begin{aligned} &[X_1,X_2]=x_5+(a^2B-2a)x_6=X_5\,,\\ &[X_1,X_3]=x_5+(d-a-adB)x_6=X_6\,,\\ &[X_2,X_4]=x_5+(d-a-adB)x_6=X_6\,,\\ &[X_3,X_4]=x_5+(d^2B+2d)x_6=\lambda X_5+(1-\lambda)X_6\,. \end{aligned} \tag{4.10}$$

Since a, d and $a+d$ are all non-zero, X_5 and X_6 are linearly independent. Straightforward computations yield

$$\lambda=\frac{dB+1}{aB-1}\neq 0\,,\ 1.$$

Thus we have the following alternative.

SUBCASE 2.3.1: $C = 1$; A, B, $4A + B^2 \neq 0$. Let now

$$X_1 = x_1 + tx_4, \quad X_2 = x_2 - tx_3, \quad X_3 = x_3, \quad X_4 = x_4, \tag{4.11}$$

where $t = -B/2A$. Hence

$$\begin{aligned} [X_1, X_2] &= (1 + t^2A)x_5 + (t^2B - 2t)x_6 = X_5, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = \alpha X_5 = \tfrac{4A^2}{4A+B^2} X_5. \end{aligned} \tag{4.12}$$

SUBCASE 2.3.2: $C = 1$; $A, B \neq 0$, $4A + B^2 = 0$. The same transformation (4.11) yields

$$\begin{aligned} [X_1, X_2] &= 0, \\ [X_1, X_3] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_2, X_4] &= -tAx_5 + (1 - tB)x_6 = X_6, \\ [X_3, X_4] &= Ax_5 + Bx_6 = X_5 \end{aligned} \tag{4.13}$$

and, after the transformation $x_1 = X_3$, $x_2 = X_4$, $x_3 = X_1$, $x_4 = X_2$, $x_5 = X_5$, $x_6 = -X_6$, we obtain (4.12) with $\alpha = 0$. Anyway, we obtain the desired form of $\mathfrak{h}$

$$[x_1, x_2] = x_5, \quad [x_1, x_3] = x_6, \quad [x_2, x_4] = x_6, \quad [x_3, x_4] = Ax_5. \tag{4.14}$$

Scaling x_3, x_4 by $\lambda \neq 0$ we may suppose that $A = m$ where m is a square-free integer as above.

In order to conclude the proof of the first part of the theorem we point out an isomorphism $\rho\colon \mathfrak{h} \to \mathfrak{hei}_3(A_m)$. Recall that A_m has a basis 1, x over $\mathbf{Q}$ such that $x^2 = m$. Here are the matrices representing $\rho(x_i)$ if $m \neq 1$ (the case $m = 1$ is left to the reader as an easy exercise):

$$\begin{aligned} &\rho(x_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\ &\rho(x_3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_4) = \begin{pmatrix} 0 & -x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \rho(x_6) = \begin{pmatrix} 0 & 0 & x \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{4.15}$$

Now it is evident that $\mathfrak{h} \otimes_{\mathbf{Q}} \mathbf{R}$ is isomorphic to either $\mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$, or $\mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$, or $\mathfrak{hei}_3(\mathbf{C})$ depending on the sign of m. Thus, we have classified up to $\mathbf{Q}$-isomorphism all rational forms for these 3 real Lie algebras. By Theorem 2 these forms are non-isomorphic. The proof of the theorem is complete. $\square$

REMARK. It is worth mentioning that the above three real Lie algebras are not pairwise isomorphic over $\mathbf{R}$. Indeed, the centralizer of any element in $\mathfrak{g}_- = \mathfrak{hei}_3(\mathbf{C})$ is even dimensional over $\mathbf{R}$ since this algebra can be viewed as a complex Lie algebra, whereas in both $\mathfrak{g}_+ = \mathfrak{hei}_3(\mathbf{R}) \oplus \mathfrak{hei}_3(\mathbf{R})$ and $\mathfrak{g}_0 = \mathfrak{hei}_3(\mathbf{R}[x]/(x^2))$ there are elements with 5-dimensional centralizers. In order to show that the last two algebras are not isomorphic we need some more information about elements with 5-dimensional centralizers.

The centralizer $C(x)$ will not be changed if we scale x by any $\lambda \neq 0$ or add to x any central element. This means that dimension of the centralizer is a well-defined function on the projective space $\mathbf{P}(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])$ where $\mathfrak{g}$ is either $\mathfrak{g}_+$ or $\mathfrak{g}_0$. Straightforward computations show that in $\mathbf{P}(\mathfrak{g}_0/[\mathfrak{g}_0,\mathfrak{g}_0])$ all points with 5-dimensional centralizer belong to a unique line whereas in $\mathbf{P}(\mathfrak{g}_+/[\mathfrak{g}_+,\mathfrak{g}_+])$ the points under consideration form two disjoint lines.

REFERENCES

[1] BOURBAKI, N. *Groupes et algèbres de Lie. Chap. 1, 3.* Hermann, Paris, 1972.

[2] CORWIN, L. J. and F. P. GREENLEAF. *Representations of Nilpotent Lie Groups and their Applications. Part I.* Cambridge Univ. Press, 1989.

[3] DIXMIER, J. Sur les représentations unitaires des groupes de Lie nilpotents III. *Canad. J. Math. 10* (1958), 321–348.

[4] RAGHUNATHAN, M. *Discrete Subgroups of Lie Groups.* Springer-Verlag, Berlin, 1972.

[5] MALCEV, A. I. On a class of homogeneous spaces. *Amer. Math. Soc. Translation 39* (1951); *Izvestiya Akad. Nauk SSSR. Ser. Mat. 13* (1949), 9–32.

(Reçu le 25 juillet 2001)

Yu. S. Semenov

MIIT, division 'Applied Mathematics – 1'
Obraztsova 15
127994 Moscow
Russia
e-mail: yury_semenov@hotmail.com