# 1. Introduction

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# *MM*-SPACES AND GROUP ACTIONS

by Vladimir PESTOV

ABSTRACT. These are introductory notes on some aspects of concentration of measure in the presence of an acting group and its links to Ramsey theory [1]).

## 1. INTRODUCTION

It can be argued that the theory we are interested in (call it theory of *mm*-spaces, the phenomenon of concentration of measure on high-dimensional structures, asymptotic geometric analysis, geometry of large dimensions, ...) has been largely shaped up by three publications. These are: the book by Paul Lévy [Lév], Vitali Milman's new proof of the Dvoretzky theorem [M1], and the paper by Gromov and Milman [Gr-M1] which had set up a framework for systematically dealing with concentration of measure. Significantly, in the two latter papers concentration goes hand in hand with group actions on suitable spaces with metric and measure.

It is also known that concentration of measure and combinatorial, Ramsey-type results have a similar nature and are often found together [M3].

A number of attempts have been made to understand the nature of the interplay between concentration, transformation groups, and Ramsey

[1]) Based on a lecture given in the framework of *Séminaire Borel de III^e Cycle romand de Mathématiques: "2001: an mm-space odyssey" (Espaces avec une métrique et une mesure, d'après M. Gromov)* at the Institute of Mathematics, University of Berne, and a *Séminaire du Lièvre* talk at the Department of Mathematics, University of Geneva. The author gratefully acknowledges generous support from the Swiss National Science Foundation during his visit in April–May 2001 and thanks Pierre de la Harpe for his hospitality and many stimulating conversations. While in Switzerland, the author has also greatly benefitted from discussions with Gulnara Arzhantseva, Anna Erschler, Thierry Giordano, Eli Glasner, Rostislav Grigorchuk, Volodymyr Nekrashevych, Vitali Milman, and Tatiana Nagnibeda. Partial support also came from the Marsden Fund of the Royal Society of New Zealand. Numerous remarks by the anonymous referee have been most helpful.

theory, cf. papers by Gromov [Gr1], Milman [M2,M3], and some others [A-M,Gl,P2,P3,G-P,Gl-W]. However, it is safe to say that there is still a long way to go towards the full understanding of the picture.

Here we aim at providing a readable introduction into this circle of ideas.

## 2. Some concepts of asymptotic geometric analysis

Definition 1. A *space with metric and measure,* or an *mm-space*, is a triple $(X, d, \mu)$, where $d$ is a metric on a set $X$ and $\mu$ is a finite Borel measure on the metric space $(X, d)$. It will be convenient to assume throughout that $\mu$ is a probability measure, that is, normalized to one.

Definition 2. The *concentration function* $\alpha_X$ of an $mm$-space $X = (X, d, \mu)$ is defined for non-negative real $\varepsilon$ as follows:

$$\alpha_X(\varepsilon) = \begin{cases} \frac{1}{2} & \text{if } \varepsilon = 0, \\ 1 - \inf\{\mu(A_\varepsilon) \colon A \subseteq X \text{ is Borel, } \mu(A) \geq \frac{1}{2}\} & \text{if } \varepsilon > 0. \end{cases}$$

Here $A_\varepsilon$ denotes the $\varepsilon$-neighbourhood ($\varepsilon$-fattening, $\varepsilon$-thickening) of $A$.

Exercise 1. Prove that $\alpha(\varepsilon) \to 0$ as $\varepsilon \to \infty$. (For spaces of finite diameter this is of course obvious.)

Definition 3. An infinite family of $mm$-spaces, $(X_n, d_n, \mu_n)_{n=1}^{\infty}$, is called a *Lévy family* if the concentration functions $\alpha_n$ of $X_n$ converge to zero pointwise on $(0, \infty)$:

$$\forall \varepsilon > 0, \ \ \alpha_n(\varepsilon) \to 0 \text{ as } n \to \infty.$$

Exercise 2. Prove that the above condition is equivalent to the following. Let $A_n \subseteq X_n$ be Borel subsets with the property that

$$\liminf_{n \to \infty} \mu_n(A_n) > 0 .$$

Then

$$\forall \varepsilon > 0, \ \ \lim_{n \to \infty} \mu_n((A_n)_\varepsilon) = 1 .$$

The following are some of the most common examples of Lévy families.