2. Some concepts of asymptotic geometric analysis

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theory, cf. papers by Gromov [Gr1], Milman [M2,M3], and some others [A-M,Gl,P2,P3,G-P,Gl-W]. However, it is safe to say that there is still a long way to go towards the full understanding of the picture.

Here we aim at providing a readable introduction into this circle of ideas.

2. Some concepts of asymptotic geometric analysis

Definition 1. A *space with metric and measure,* or an *mm-space*, is a triple (X, d, μ), where d is a metric on a set X and μ is a finite Borel measure on the metric space (X, d). It will be convenient to assume throughout that μ is a probability measure, that is, normalized to one.

Definition 2. The *concentration function* α_X of an mm-space $X = (X, d, \mu)$ is defined for non-negative real ε as follows:

$$\alpha_X(\varepsilon) = \begin{cases} \frac{1}{2} & \text{if } \varepsilon = 0, \\ 1 - \inf\{\mu(A_\varepsilon) \colon A \subseteq X \text{ is Borel, } \mu(A) \geq \frac{1}{2}\} & \text{if } \varepsilon > 0. \end{cases}$$

Here A_ε denotes the ε-neighbourhood (ε-fattening, ε-thickening) of A.

Exercise 1. Prove that $\alpha(\varepsilon) \to 0$ as $\varepsilon \to \infty$. (For spaces of finite diameter this is of course obvious.)

Definition 3. An infinite family of mm-spaces, $(X_n, d_n, \mu_n)_{n=1}^{\infty}$, is called a *Lévy family* if the concentration functions α_n of X_n converge to zero pointwise on $(0, \infty)$:

$$\forall \varepsilon > 0, \ \alpha_n(\varepsilon) \to 0 \text{ as } n \to \infty.$$

Exercise 2. Prove that the above condition is equivalent to the following. Let $A_n \subseteq X_n$ be Borel subsets with the property that

$$\liminf_{n \to \infty} \mu_n(A_n) > 0 .$$

Then

$$\forall \varepsilon > 0, \ \lim_{n \to \infty} \mu_n((A_n)_\varepsilon) = 1 .$$

The following are some of the most common examples of Lévy families.

EXAMPLE 1. Unit spheres $\mathbf{S}^n$ in the Euclidean spaces $\mathbf{R}^{n+1}$, equipped with the Euclidean (or geodesic) distances and the normalized Haar measures (that is, the unique rotation-invariant probability measures). This result is due to Paul Lévy [Lév], though his proof, based on the isoperimetric inequality, was only made rigorous much later by Gromov [Gr2]. (Nowadays simpler proofs, using the Brunn–Minkowski inequality, are known, cf. [Gr-M2, Sch].)

EXAMPLE 2. The special orthogonal groups $\mathrm{SO}(n)$, equipped with the normalized Haar measure and the uniform operator metric,

$$d(T,S) := \|T - S\|,$$

induced from $\mathcal{B}(\mathbf{R}^n) \cong M_n$. This was established by Gromov and Milman [Gr-M1]. The same argument holds for the special unitary groups.

EXAMPLE 3. The family of finite permutation groups (S_n), equipped with the uniform (normalized counting) measure and the Hamming distance:

$$d(\sigma,\tau) = \frac{1}{n}\left|\{i\colon \sigma(i) \neq \tau(i)\}\right|.$$

The result is due to Maurey [Ma], see also [Ta1].

EXAMPLE 4. The Hamming cubes $\{0,1\}^n$ equipped with the normalized counting measure and the Hamming distance $d(x,y) = \frac{1}{n}\left|\{i\colon x_i \neq y_i\}\right|$ form a Lévy family [Sch,M-S].

REMARK 1. All of the above are *normal* Lévy families, meaning that the concentration functions α_n admit Gaussian upper bounds:

$$\alpha_n(\varepsilon) \leq C_1 \exp(-C_2 n\varepsilon^2)$$

for some $C_1, C_2 > 0$.

It should be noted that this is not always the case for ‘naturally occurring’ Lévy families. For instance, the groups $\mathrm{SL}(2,\mathbf{F}_p)$, where p are prime numbers, equipped with the normalized counting measure and the word metric given by a fixed system of generators in $\mathrm{SL}(2,\mathbf{Z})$, form a Lévy family with $\alpha_p(\varepsilon) \leq C_1 \exp(-C_2\sqrt{p}\,\varepsilon)$, [A-M, M4]. (Recall in this connection that the n-th prime number $p_n \sim n\log n$.)

REMARK 2. In Example 4, replace $\{0,1\}$ with any probability measure space, $X = (X,\mu)$. Equip every finite power X^n with the product measure $\mu^{\otimes n}$ and the normalized Hamming distance $d(x,y) = \frac{1}{n}|\{i\colon x_i \neq y_i\}|$. Unless X is purely atomic, the measures $\mu^{\otimes n}$ are not Borel, and thus X^n aren't even mm-spaces in the sense of our definition. At the same time, if in the definition of the concentration function we only restrict ourselves to measurable subsets A such that A_ε are also measurable, it can be shown that $X^n, n \in \mathbf{N}$ form a Lévy family in a very reasonable sense. (See [Ta1,Ta3] for far-reaching variations.) If anything, this shows that the full formalization of the subject has not yet been achieved and nothing is cast in stone.

Notice that the mm-spaces from the above Examples 1–4 are at the same time (phase spaces of) topological transformation groups, with both metrics and measures being invariant under group actions. In Example 1 it is the action of the orthogonal — or the unitary — group on the sphere, while in Examples 2–4 the groups act upon themselves on the left.

3. A TRANSFORMATION GROUP FRAMEWORK

Here is the idea of what kind of interaction between concentration phenomenon and group actions one should expect. The following example is borrowed from a paper by Vitali Milman [M4].

Suppose a group G acts on an mm-space (X,d,μ) by measure-preserving isometries. Assume that the mm-space X strongly concentrates, that is, the function $\alpha_X(\varepsilon)$ drops off sharply already for small values of ε. Let us assume, for instance, that the concentration is so strong that, whenever $\mu(A) \geq \frac{1}{7}$, the measure of the $\frac{1}{10}$-neighbourhood of A is strictly greater than 0.99. (Cf. Exercise 2.)

If now we partition X into seven pieces, and pick at random one hundred elements $g_1, g_2, \dots, g_{100} \in G$, then at least one of the pieces, say A, has the property that all one hundred translates, of $\frac{1}{10}$-neighbourhoods of A by our elements g_i have a point, x^*, in common. Equivalently, x^* is 'close' (closer than $\frac{1}{10}$) to each of the one hundred translates of A.

The above effect becomes more pronounced the higher the level of concentration is. Partition a concentrated ('high-dimensional') mm-space into a small number of subsets, and at least one of them is hard to move.

In order to set up a formal framework, we assume all topological spaces and topological groups appearing in this article to be metrizable, for the reasons