6. Ramsey-Dvoretzky-Milman property

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6. Ramsey–Dvoretzky–Milman property

6.1 Extreme amenability and small oscillations

One way to intuitively describe a 'Ramsey-type result' is as follows. Suppose $\mathfrak{X}$ is a large (and often highly homogeneous) structure of some sort or other. Let $\mathfrak{X}$ be partitioned into a finite number of pieces in an arbitrary way. No matter how irregular and 'ragged' the pieces are, at least one of them always contains the remnants of the original structure, that is, a (possibly much smaller, but still detectable) substructure of the same type which survived intact.

We are now going to link explicitly the fixed point on compacta property to Ramsey-type results. Here is the first step.

Exercise 8. Prove that a topological group G is extremely amenable if and only if for every finite collection $g_1, \ldots, g_n$ of elements of G, every bounded right uniformly continuous function $f\colon G \to \mathbf{R}^N$ from G to a finite-dimensional Euclidean space, and every $\varepsilon > 0$ there is an $h \in G$ such that $|f(h) - f(g_i h)| < \varepsilon$ for each $i = 1, 2, \ldots, n$.

[*Hints.* ($\Rightarrow$) The action of G on the space $\mathcal{S}(G)$ of maximal ideals of the C^*-algebra $\mathrm{RUCB}(G)$ is continuous, and G itself can be thought of as an everywhere dense subset of $\mathcal{S}(G)$.

($\Leftarrow$) Form a net of suitably indexed elements h as above and consider any limit point of the net $h_\alpha \cdot \xi$, where ξ is an arbitrary element of the compact space upon which G acts continuously.]

Exercise 9. Prove that the above condition for extreme amenability is, in turn, equivalent to the following. For every bounded *left* uniformly continuous function f from G to a finite-dimensional Euclidean space, every finite subset F of G, and every $\varepsilon > 0$, the oscillation of f on a suitable left translate of F is less than ε:

$$\exists g \in G,\ \mathrm{Osc}(f|_{gF}) < \varepsilon\,.$$

It is convenient to deal with the above property in a more general context of G-spaces.

DEFINITION 11 [Gr1]. Say that a G-space X (in our agreed sense) has the *Ramsey–Dvoretzky–Milman property* if for every bounded uniformly continuous function f from X to a finite-dimensional Euclidean space, every $\varepsilon > 0$, and every finite $F \subseteq X$, there is a $g \in G$ with the property

$$\mathrm{Osc}(f|_{gF}) < \varepsilon\,.$$

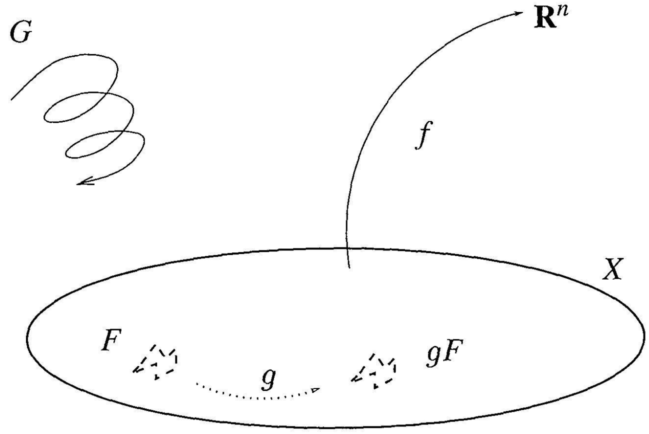

FIGURE 3
The Ramsey–Dvoretzky–Milman property

REMARK 9. Equivalently, F can be assumed compact.

COROLLARY 7. *For a topological group G the following are equivalent:*
(i) *G is extremely amenable;*
(ii) *every metric space X upon which G acts continuously and transitively by isometries has the R–D–M property;*
(iii) *every homogeneous factor-space G/H, equipped with a left-invariant metric (or the left uniform structure), has the R–D–M property.*

Next, we will discover two very important situations where the R–D–M property appears naturally.

6.2 DVORETZKY'S THEOREM

Here is the famous result.

THEOREM (Arieh Dvoretzky). *For all $\varepsilon > 0$ there is a constant $c = c(\varepsilon) > 0$ such that for any n-dimensional normed space $(X, \|\cdot\|_E)$ there is a subspace V of $\dim V \geq c \log n$ and a Euclidean norm $\|\cdot\|_2$ with $\|x\|_2 \leq \|x\|_E \leq (1+\varepsilon)\|x\|_2$ for all $x \in V$.*

The studies of the phenomenon of concentration of measure were given a boost by Vitali Milman's new proof of the Dvoretzky theorem [M1], based on a suitable finite-dimensional approximation to the lemma which follows directly from results that we have previously stated:

LEMMA (Milman). *The pair* $(\mathrm{U}(\mathcal{H}), \mathbf{S}^\infty)$ *has the R–D–M property, where* $\mathbf{S}^\infty$ *is the unit sphere of an infinite-dimensional Hilbert space* $\mathcal{H}$.

6.3 RAMSEY'S THEOREM

Let r be a positive natural number. By $[r]$ one denotes the set $\{1,2,\dots,r\}$. A *colouring* of a set X with r colours, or simply r*-colouring,* is any map $\chi\colon X \to [r]$. A subset $A \subseteq X$ is *monochromatic* if for every $a,b \in A$ one has $\chi(a) = \chi(b)$.

Put otherwise, a finite colouring of a set X is nothing but a partition of X into finitely many (disjoint) subsets.

Let X be a set, and let k be a natural number. Denote by $[X]^k$ the set of all k-subsets of X, that is, all (unordered !) subsets containing exactly k elements.

INFINITE RAMSEY THEOREM. *Let* X *be an infinite set, and let* k *be a natural number. For every finite colouring of* $[X]^k$ *there exists an infinite subset* $A \subseteq X$ *such that the set* $[A]^k$ *is monochromatic.*

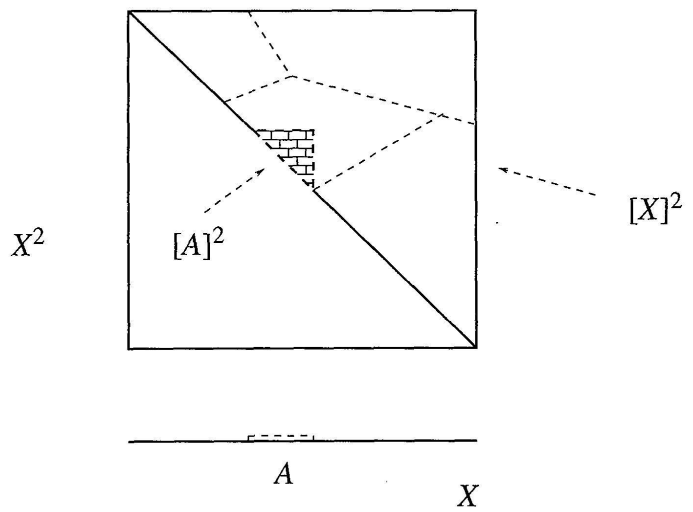

FIGURE 4
Ramsey theorem for $k = 2$

REMARK 10. For $k = 1$ the statement is simply the pigeonhole principle. Here is a popular interpretation of the result in the case $k = 2$. Among infinitely many people, either there is an infinite subset of people every two of whom know each other, or there is an infinite subset no two members of which know each other.

FINITE RAMSEY THEOREM. *For every triple of natural numbers, k, l, r, there exists a natural number $R(k, l, r)$ with the following property. If $N \geq R(k, l, r)$ and the set of all k-subsets of $[N]$ is coloured using r colours, then there is a subset $A \subseteq [N]$ of cardinality $|A| = l$ such that all k-subsets of A have the same colour.*

REMARK 11. The Infinite Ramsey Theorem implies the finite version through a simple compactness argument. At the same time, the infinite version does not seem quite to follow from the finite one. The finite version is equivalent to the following statement:

Let X be an infinite set, and let k be a natural number. For every finite colouring of $[X]^k$ and every natural n there exists a subset $A \subseteq X$ of cardinality n such that $[A]^k$ is monochromatic.

A good introductory reference to Ramsey theory is [Gra].

Denote by $\mathrm{Aut}(\mathbf{Q})$ the group of all order-preserving bijections of the set of rational numbers, equipped with the topology of pointwise convergence on the discrete set $\mathbf{Q}$. In other words, we regard $\mathrm{Aut}(\mathbf{Q})$ as a (closed) topological subgroup of S_∞. A basic system of neighbourhoods of identity is formed by open subgroups each of which stabilizes elements of a given finite subset of $\mathbf{Q}$.

EXERCISE 10. Use Corollary 7 to prove that the finite Ramsey theorem is equivalent to the statement:

The topological group $\mathrm{Aut}(\mathbf{Q})$ *is extremely amenable.*

[*Hint.* For a finite subset $M \subset \mathbf{Q}$, the left factor space of $\mathrm{Aut}(\mathbf{Q})$ by the stabilizer of M can be identified with the set $[\mathbf{Q}]^n$, where $n = |M|$, equipped with the discrete uniformity (or $\{0,1\}$-valued metric). Cover $[\mathbf{Q}]^n$

with finitely many sets on each of which the given function f has oscillation $< \varepsilon$, and apply Ramsey's theorem. Use Remark 11.]

6.4 Extreme amenability and minimal flows

Corollary 8. *The group of orientation-preserving homeomorphisms of the closed unit interval,* $\mathrm{Homeo}_+(\mathbf{I})$*, equipped with the compact-open topology, is extremely amenable.*

Proof. Indeed, the extremely amenable group $\mathrm{Aut}(\mathbf{Q})$ admits a continuous monomorphism with a dense image into the group $\mathrm{Homeo}_+(\mathbf{I})$.

Remark 12. Thompson's group F consists of all piecewise-linear homeomorphisms of the interval whose points of non-smoothness are finitely many dyadic rational numbers, and the slopes of any linear part are powers of 2. (See [CFP].) It is a major open question in combinatorial group theory whether the Thompson group is amenable. Since F is everywhere dense in $\mathrm{Homeo}_+(\mathbf{I})$, our Corollary 8 does not contradict the possible amenability of F.

Using the extreme amenability of the topological groups $\mathrm{Aut}(\mathbf{Q})$ and $\mathrm{Homeo}_+(\mathbf{I})$, one is able to compute explicitly the universal minimal flows of some larger topological groups as follows.

Corollary 9. *The circle* $\mathbf{S}^1$ *forms the universal minimal* $\mathrm{Homeo}_+(\mathbf{S}^1)$*-space.*

Proof. Let $\theta \in \mathbf{S}^1$ be an arbitrary point. The isotropy subgroup St_θ of θ is isomorphic to $\mathrm{Homeo}_+(\mathbf{I})$. Because of that, whenever the topological group $\mathrm{Homeo}_+(\mathbf{S}^1)$ acts continuously on a compact space X, the subgroup St_θ has a fixed point, say $x' \in X$. The mapping $\mathrm{Homeo}_+(\mathbf{S}^1) \ni h \mapsto h(x') \in X$ is constant on the left St_θ-cosets and therefore gives rise to a continuous equivariant map $\mathrm{Homeo}_+(\mathbf{S}^1)/\mathrm{St}_\theta \cong \mathbf{S}^1 \to X$.

For the above results concerning groups $\mathrm{Aut}(\mathbf{Q})$, $\mathrm{Homeo}_+(\mathbf{I})$, and $\mathrm{Homeo}_+(\mathbf{S}^1)$, see [P1].

Now denote by LO the set of all linear orders on $\mathbf{Z}$, equipped with the (compact) topology induced from $\{0,1\}^{\mathbf{Z}\times\mathbf{Z}}$. The group S_∞ acts on LO by double permutations.

EXERCISE 11. Prove that the action of S_∞ on LO is continuous and minimal (that is, the orbit of each linear order is everywhere dense in LO).

Recall that a linear order $\prec$ is called *dense* if it has no gaps. A dense linear order without least and greatest elements is said to be of type η. The collection LO_η of all linear orders of type η on $\mathbf{Z}$ can be identified with the factor space $S_\infty/\operatorname{Aut}(\prec)$ through the correspondence $\sigma \mapsto {}^\sigma\!\prec$. Here $\prec$ is some chosen linear order of type η on $\mathbf{Z}$ and $\operatorname{Aut}(\prec)$ stands for the group of order-preserving self-bijections of $(\mathbf{Z}, \prec)$, acting on the space of orders in a natural way: $(x \ {}^\sigma\!\prec y) \Leftrightarrow \sigma^{-1}x \prec \sigma^{-1}y$.

EXERCISE 12. Show that under the above identification the uniform structure on LO_η, induced from the compact space LO, is the finest uniform structure making the quotient map $S_\infty \to S_\infty/\operatorname{Aut}(\prec) \cong \mathrm{LO}_\eta$ right uniformly continuous.

Let now X be a compact S_∞-space. The topological subgroup $\operatorname{Aut}(\prec)$ of S_∞ has a fixed point in X, say x' (Exercise 10). The mapping $S_\infty \ni \sigma \mapsto \sigma(x') \in X$ is constant on the left $\operatorname{Aut}(\prec)$-cosets and thus gives rise to a mapping $\varphi\colon \mathrm{LO}_\eta \to X$. Using Exercise 12, it is easy to see that φ is right uniformly continuous and thus extends to a morphism of S_∞-spaces $\mathrm{LO} \to X$. We have established the following result.

THEOREM 6 (Glasner and Weiss [Gl-W]). *The compact space* LO *forms the universal minimal* S_∞*-space.*

6.5 THE URYSOHN METRIC SPACE

The *universal Urysohn metric space* $\mathbf{U}$ [Ur] is determined uniquely (up to an isometry) by the following conditions:

(i) $\mathbf{U}$ is a complete separable metric space;

(ii) $\mathbf{U}$ is ω-homogeneous, that is, every isometry between two finite subspaces of $\mathbf{U}$ extends to an isometry of $\mathbf{U}$;

(iii) $\mathbf{U}$ contains an isometric copy of every separable metric space.

A probabilistic description of this space was given by Vershik [Ver]: the completion of the space of integers equipped with a 'sufficiently random' metric is almost surely isometric to $\mathbf{U}$.

The group of isometries $\operatorname{Iso}(\mathbf{U})$ with the compact-open topology is a Polish (complete metric separable) topological group, which also possesses a universality property: it contains an isomorphic copy of every separable metric group [Usp]. See also [Gr3].

Using concentration of measure, one can prove that the group Iso$(\mathbf{U})$ is extremely amenable. The Ramsey–Dvoretzky–Milman property leads to the following Ramsey-type result:

Let F be a finite metric space, and let all isometric embeddings of F into $\mathbf{U}$ be coloured using finitely many colours. Then for every finite metric space G and every $\varepsilon > 0$ there is an isometric copy $G' \subset \mathbf{U}$ of G such that all isometric embeddings of F into $\mathbf{U}$ that factor through G are monochromatic to within ε.

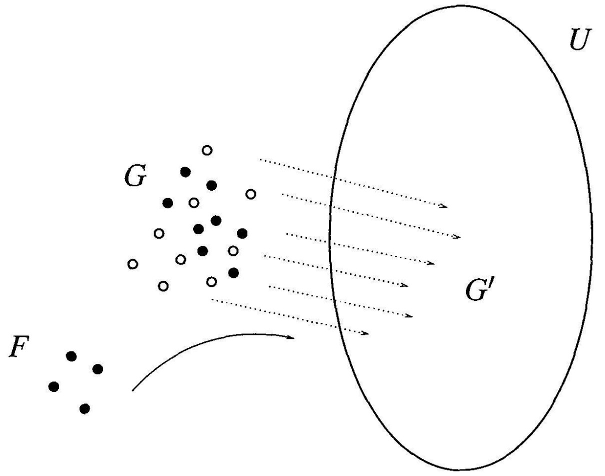

FIGURE 5
A Ramsey-type result for metric spaces

Here we say that a set A is *monochromatic to within* ε if there is a monochromatic set A' at a Hausdorff distance $< \varepsilon$ from A. In our case, the Hausdorff distance is formed with regard to the uniform metric on $\mathbf{U}^F$.

One can also obtain similar results, for example, for the separable Hilbert space ℓ_2 and for the unit sphere $\mathbf{S}^\infty$ in ℓ_2 [P3].

7. CONCENTRATION TO A NON-TRIVIAL SPACE

Let f be a Borel measurable real-valued function on an *mm*-space $X = (X, d, \mu)$. A number $M = M_f$ is called a *median* (or *Lévy mean*) of f if both $f^{-1}[M, +\infty)$ and $f^{-1}(-\infty, M]$ have measure $\geq \frac{1}{2}$.

EXERCISE 13. Show that the median M_f always exists, though it need not be unique.