HOLONOMY AND SUBMANIFOLD GEOMETRY

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HOLONOMY AND SUBMANIFOLD GEOMETRY

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ABSTRACT. We survey some applications of holonomic methods to the study of submanifold geometry, showing the consequences of some sort of extrinsic version of the de Rham Decomposition Theorem and of Berger's Theorem: the so-called *Normal Holonomy Theorem*. At the same time, from geometric methods in submanifold theory we sketch some very strong applications to the holonomy of Lorentzian manifolds. Moreover we give a conceptual modern proof of a result of Kostant for homogeneous spaces.

1. INTRODUCTION

A connection on a connected Riemannian manifold M can be interpreted as a way of comparing tangent spaces at different points, by means of parallel transport.

The parallel translation depends, in general, on the curve joining two points and this dependence is measured by the *holonomy group*, i.e. the linear group of isometries obtained by parallel transporting along based loops.

Actually holonomy groups can be defined for any connection on a vector bundle. For example, in this note we will be particularly interested in the holonomy group of the normal connection, called *normal holonomy group*.

Holonomy plays an important role in (intrinsic) Riemannian geometry, in the context of special Riemannian metrics, e.g., symmetric, Kähler, hyperkähler and quaternionic Kähler metrics.

The main purpose of this note is to survey the application of holonomic methods to the study of submanifold geometry and vice versa. Namely, we will sketch some very strong applications of geometric methods in submanifold

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theory to the holonomy of Lorentzian manifolds. But we will also be interested in Riemannian holonomy and will give a conceptual modern proof of a result of Kostant for homogeneous spaces.

The survey is organized as follows. In Section 2 we recall some important results on holonomy for a Riemannian manifold. This also allows one to make a comparison with results on normal holonomy, to which Section 3 is devoted. The crucial result in the extrinsic geometry is the Normal Holonomy Theorem (3.2) [O1], which asserts that the non trivial part of normal holonomy acting on the normal space is an s-representation (i.e., isotropy representation of a Riemannian symmetric space). Recall that, by Dadok's Theorem, s-representations are orbit equivalent to *polar actions* and that principal orbits of s-representation are *isoparametric submanifolds*. The Normal Holonomy Theorem is somehow an extrinsic analogue of the de Rham Decomposition Theorem and of Berger's Theorem on Riemannian holonomy. One of its main consequences is the recognition that orbits of s-representations play a similar role, in submanifold geometry, as Riemannian symmetric spaces in intrinsic Riemannian geometry. This is illustrated by high rank theorems (Theorems 3.6, 3.9, 3.10), which have similarities with higher rank results on Riemannian manifolds. The extrinsic notion of rank is linked to maximal flat parallel subbundles of the normal bundle.

In Section 4 we relate homogeneity and holonomy in the general framework of homogeneous (pseudo) Riemannian vector bundles endowed with a connection. The Lie algebra of the holonomy group (holonomy algebra) can be described in terms of projection of Killing vector fields on the homogeneous bundle. As an application to Riemannian manifolds we get Kostant's method for computing the Lie algebra of the holonomy group of a homogeneous Riemannian manifold. Moreover we obtain a local characterization of Kähler and Ricci flat Riemannian manifolds in terms of the normalizer of the Lie algebra of the local holonomy group (Proposition 4.1). For a submanifold M which is an orbit of an orthogonal representation of a Lie group G, normal holonomy measures how much G fails to act polarly and how much M fails to be a principal orbit. Polar actions on the tangent bundle of a simply connected Riemannian manifold M allow to characterize symmetric spaces. This is done in Theorem 4.2, which states that the tangent bundle TM admits a polar action having M as an orbit if and only if M is symmetric.

In Section 5 we show how the theory of homogeneous submanifolds of the hyperbolic space H^n can be used to obtain general results on the action of a connected Lie subgroup of $\mathrm{O}(n,1)$ on the Lorentzian space $\mathbf{R}^{n,1}$. A consequence is a completely geometric proof, using submanifold geometry, of the fact that the restricted holonomy group of an irreducible Lorentzian manifold is $SO_0(n, 1)$, [B1], [B2].

At least a sketch of proof is given for every result mentioned. In some cases we include complete proofs if they are hard to find in the literature.

2. Riemannian holonomy

We first recall some basic facts on holonomy. For more details we refer to [Be, Chapter 10]. Let M be a Riemannian manifold, which we will always assume to be connected. If we fix a point $p \in M$, the parallel transport τ_γ along any loop γ at p determines an isometry of T_pM. The set of all such isometries is a subgroup $\Phi_p(M)$ of the orthogonal group $O(T_p(M))$, called the *holonomy subgroup* of M at p. If q is another point of M, β a path from p to q and τ_β the parallel transport along β, we have $\Phi_q(M) = \tau_\beta \, \Phi_p(M) \, \tau_\beta^{-1}$, so that holonomy groups at different points are conjugated and one speaks of holonomy group of M neglecting the base point. There is a variant of this definition, the *restricted* holonomy group $\Phi_p^*(M)$, obtained by considering only those loops which are homotopically trivial. This group actually behaves more nicely: it is a connected, closed Lie subgroup of $SO(T_pM)$ and is in fact the identity component of $\Phi_p(M)$. It can be regarded as the holonomy group of the universal covering space of M.

Holonomy is closely tied to curvature, which is roughly an infinitesimal measure of holonomy. More precisely, the Ambrose-Singer Holonomy Theorem states that the Lie algebra of the holonomy group is spanned by the curvature operators R_{xy}, $x, y \in T_pM$, together with their parallel translates along loops at p.

In order to describe the importance holonomy plays in intrinsic geometry, we recall an important property of holonomy, the so-called *holonomy principle*: evaluation at p establishes a one-to-one correspondence between parallel tensor fields and tensors invariant under holonomy. The existence of holonomy invariant tensors has strong consequences on the geometry. Here are some examples of this situation.

- For a generic metric $\Phi_p(M) = O(T_pM)$, and there is no invariant tensor besides those generated by the metric (see e.g. [Iw]).

- The existence of an invariant subspace implies that the manifold locally splits (de Rham Decomposition Theorem). Thus one can always restrict attention to irreducible holonomy actions.

• It is a classical result of Cartan that, if the Riemannian curvature tensor of a Riemannian manifold M is invariant under parallel transport, M is locally symmetric, i.e., at each point p in M there exists an open ball $B_r(p)$ such that the corresponding local geodesic symmetry s_p is an isometry. A Riemannian manifold is called a *symmetric space* if at each point $p \in M$ such a local geodesic symmetry extends to a global isometry $s_p \colon M \to M$. Symmetric spaces play a prominent role in Riemannian geometry and are very tightly connected to holonomy.

Indeed, let M be an irreducible symmetric space, which can be represented as a quotient $M = G/K$, where G is the identity component of the isometry group of M and K is the isotropy subgroup at some point $p \in M$. One can show that the isotropy representation of K on T_pM agrees with the (effective) representation of the restricted holonomy group $\Phi^*_p(M)$ on T_pM. Observe that, by the Ambrose-Singer Holonomy Theorem and the invariance of the curvature tensor by parallel transport, the holonomy algebra is spanned by the curvature operators R_{xy}, $x, y \in T_pM$. Now the curvature operators allow to recover the symmetric space by a classical construction due to E. Cartan. We briefly outline this construction, which can actually be carried out for any algebraic curvature tensor on some vector space V (i.e., a tensor with the same algebraic properties as the curvature tensor, including the first Bianchi identity) which is invariant by the action of a group K (i.e. $k \cdot R = R$, for any $k \in K$). Indeed, one can construct an orthogonal symmetric Lie algebra $\mathfrak{g}$, by setting $\mathfrak{g} := \mathfrak{k} \oplus V$ and defining

$$\begin{aligned}
[B, C] &= BC - CB, && B, C \in \mathfrak{k},\\
[x, y] &= R_{xy}, && x, y \in V,\\
[A, z] &= Az, && A \in \mathfrak{k},\ z \in V.
\end{aligned}$$

Passing to Lie groups one locally recovers G/K (globally if G/K is simply connected).

Yet another characterization of symmetric spaces in terms of holonomy is the following. One can define the *transvection group* of a Riemannian manifold N as the group $\mathrm{Tr}(N)$ of isometries of N that preserve any holonomy subbundle $\mathrm{Hol}_v N$, $v \in T_pN$. Recall that $\mathrm{Hol}_v N$ is the subset, which turns out to be a subbundle, of the tangent bundle TN obtained by parallel transport of v along any piecewise differentiable curve starting from p. More concretely, $\mathrm{Tr}(N)$ is the group of all isometries φ such that, for any $p \in N$, there exists a piecewise differentiable curve γ joining p and $\varphi(p)$ such that $\varphi_{*p} \colon T_pN \to T_{\varphi(p)}N$ coincides with the parallel transport along γ.

Now, a symmetric space M can be characterized by the fact that the transvection group acts transitively on any holonomy subbundle. This is to say that, for any $p, q \in M$, for any piecewise differentiable curve γ from p to q, there exists an isometry g such that $g(p) = q$ and $g_{*p}\colon T_pM \to T_qM$ coincides with the parallel transport along γ.

- If the Ricci tensor is parallel, then M is a product of Einstein manifolds (see e.g. [Be]).
- If there is a complex structure J on a Riemannian manifold M which is orthogonal and parallel, then M is a *Kähler* manifold. In this case the holonomy group is contained in the unitary group $\mathrm{U}(T_pM)$.

Thus, the existence of a geometric structure on a Riemannian manifold can be read in terms of the holonomy invariance of a tensor and this in turn implies a reduction of the holonomy group to a proper subgroup of $\mathrm{O}(T_pM)$.

A fundamental result for the restricted holonomy group $\Phi_p^*(M)$ of a Riemannian manifold is Berger's Theorem ([B1], see also [Be], [Sal], [Sim]), which classifies the possible irreducible actions of the restricted holonomy group on the tangent space at any point: $\Phi_p^*(M)$ is either transitive on the unit sphere of T_pM or it acts as the isotropy representation of a symmetric space (which is also called s-representation) and M is in fact locally symmetric. If the action of $\Phi_p^*(M)$ is transitive on the unit sphere of T_pM and is irreducible, then $\Phi_p^*(M)$ is one the following groups: $\mathrm{SO}(n)$, $\mathrm{U}(n/2)$ $(n \geq 4)$, $\mathrm{SU}(n/2)$ $(n \geq 4)$, $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n/4)$ $(n \geq 4)$, $\mathrm{Sp}(n/4)$ $(n \geq 4)$, $\mathrm{Spin}(9)$ $(n = 16)$, $\mathrm{Spin}(7)$ $(n = 8)$ or G_2 $(n = 7)$.

The reduction of $\Phi_p^*(M)$ to any of the above groups corresponds to some geometric structure on M.

3. Normal holonomy

Let M be a submanifold of a space of constant curvature $\overline{M}$. We will denote by α its second fundamental form, by A its shape operator sending each normal vector ξ to the self adjoint endomorphism A_ξ of the tangent space T_pM, with $\langle A_\xi x, y\rangle = \langle \alpha(x,y), \xi\rangle$ $(x, y \in T_pM)$ and by νM the normal bundle, endowed with the normal connection $\nabla^\perp$. We denote by ${\Phi^\perp}_p$ the *normal holonomy* at $p \in M$, i.e. the holonomy at p of the normal connection.

Recall that there are mainly two possible reductions for submanifolds of spaces of constant curvature. If a submanifold M of $\overline{M}$ is contained in a totally geodesic submanifold N of $\overline{M}$, then one can regard it as a submanifold of

N and one speaks of reduction of codimension for M. If M does not admit any reduction of codimension, or equivalently if M is not contained in any totally geodesic submanifold of $\bar{M}$, one calls M a *full* submanifold of $\bar{M}$. Another reduction is extrinsic splitting as a product of submanifolds and if a submanifold does not split extrinsically, it is called *irreducible.*

Like Riemannian holonomy, normal holonomy plays a fundamental role in the geometry of submanifolds of spaces of constant curvature.

We discuss the analogies and the differences with the intrinsic case in relation with the existence of holonomy invariant tensors.

- $\Phi^{\perp}{}_p = \mathrm{O}(\nu_p M)$ means that there are no invariant tensors on $\nu_p M$ besides those generated by the metric, like in the intrinsic case. This is the case of a generic submanifold.

- The existence of an invariant subspace for the normal holonomy $\Phi^{\perp}{}_p$ does not imply in general that the submanifold locally splits (both extrinsically and intrinsically). For example, for a submanifold of Euclidean space contained in a sphere, the line determined by the position vector p is an invariant subspace under normal holonomy (it always belongs to the flat part of νM), but such a submanifold does not necessarily split. However, if M is a complex submanifold of $\mathbb{C}^n$, then one has a version of the de Rham Decomposition Theorem: if $\Phi^{\perp}{}_p$ splits, M locally splits as a product of submanifolds [D2].

- In the extrinsic case several tensors play an analogous role to the Riemannian curvature tensor: the second fundamental form α (or equivalently the shape operator A) and the normal curvature $R^{\perp}$ are perhaps the most important. A problem, in the context of normal holonomy, is that these tensors do not take values in the normal spaces only. For this reason, to find holonomy invariant tensors, one has to construct new ones. An important class of tensors with values in the normal spaces only is given by the so-called *higher order mean curvatures* [St]. The mean curvature of order k in direction ξ, $H_k(\xi)$, is the k-th elementary symmetric function of the eigenvalues of the shape operator A_ξ in direction ξ. So, up to a constant, $H_k(\xi)$ is the sum of the k-th powers of the eigenvalues $\{\lambda_i\}$ of A_ξ, i.e., $\sum_{i_1<\cdots<i_k} \lambda_{i_1} \ldots \lambda_{i_k}$. Observe that $H_1(\xi) = \langle H, \xi \rangle$, where H is the mean curvature vector field. Let $h_k(\xi_1, \ldots, \xi_k)$ be the symmetric tensor on νM defined by polarization of $H_k(\xi)$. Suppose that any h_k (or equivalently any H_k) is invariant by parallel transport. Thus $H_k(\xi(t))$ is constant for any parallel normal vector field $\xi(t)$ along any piecewise differentiable curve. Since the sum of the k-th powers of the eigenvalues up to order $m = \dim M$ generate all symmetric polynomials on $\lambda_1, \ldots, \lambda_m$, the characteristic polynomial of $A_{\xi(t)}$ is constant,

so $A_{\xi(t)}$ has constant eigenvalues. Conversely, it is easy to see that if $A_{\xi(t)}$ has constant eigenvalues, any h_k (or equivalently any H_k) is invariant by normal holonomy. A submanifold with this property is called a *submanifold with constant principal curvatures.* The importance of the above *tensorial* definition of a submanifold with constant principal curvatures is illustrated in the proof of Theorem 3.3. This class of submanifolds can be regarded for many reasons (which we will clarify in this note) as the extrinsic analogue of locally symmetric spaces. A very important example of submanifolds with constant principal curvatures is given by the *orbits of s-representations*, i.e. the orbits of the isotropy representations of Riemannian symmetric spaces on their tangent spaces, which play the same role, in submanifold geometry, as symmetric spaces in Riemannian geometry (as we will illustrate later). Orbits of s-representations are intrinsically real partial flag manifolds and classically are called R-spaces.

An important special class of submanifolds with constant principal curvatures is given by those with flat normal bundle, called *isoparametric submanifolds.* Actually they are somehow "generic" among submanifolds with constant principal curvatures. Indeed E. Heintze, C. Olmos and G. Thorbergsson gave the following complete characterization of the submanifolds of space forms with constant principal curvatures.

THEOREM 3.1 ([HOT]). *Let M be a submanifold of a space form. Then M has constant principal curvatures if and only if it is either isoparametric or a focal manifold of an isoparametric submanifold.*

We will define focal manifolds later in this section.

In analogy with what we did in the intrinsic case, for symmetric spaces, one can give a characterization of submanifolds with constant principal curvatures in terms of normal holonomy. Let $\mathrm{Tr}(\nabla^{\perp})$ be the transvection group of the normal holonomy. Then the orbits of s-representations can be characterized by the fact that $\mathrm{Tr}(\nabla^{\perp})$ acts transitively on any normal holonomy subbundle. More explicitly, M is an orbit of an s-representation if and only if for any $p, q \in M$ and any curve γ on M joining p and q, there exists an isometry g of Euclidean space, leaving the submanifold M invariant, sending p to q and such that

$$g_{*p|\nu_p M} : \nu_p M \longrightarrow \nu_q M$$

coincides with the $\nabla^{\perp}$-parallel transport along γ [OS].

One can classify the behaviour of normal holonomy action on $\nu_p M$. The starting point for this is a proof of Berger's Theorem, due to J. Simons [Sim], which is based on some algebraic properties of the curvature tensor, namely its antisymmetry properties and the first Bianchi identity.

By similar methods an analogous result was proved in [O1] for the restricted normal holonomy group. Roughly, the non trivial part of normal holonomy acts on $\nu_p M$ as an s-representation. More precisely:

THEOREM 3.2 (NORMAL HOLONOMY THEOREM). *Let M be a submanifold of a space form. Let $p \in M$ and let $\Phi^{\perp *}$ be the restricted normal holonomy group at p. Then $\Phi^{\perp *}$ is compact, there exists a unique (up to order) orthogonal decomposition $\nu_p M = V_0 \oplus \cdots \oplus V_k$ of the normal space at p into $\Phi^{\perp *}$-invariant subspaces and there exist normal subgroups of $\Phi^{\perp *}$, $\Phi^{\perp}{}_0 = \{1\}, \Phi^{\perp}{}_1, \dots, \Phi^{\perp}{}_k$ such that*

(i) $\Phi^{\perp *} = \{1\} \times \Phi^{\perp}{}_1 \times \cdots \times \Phi^{\perp}{}_k$ *(direct product),*

(ii) $\Phi^{\perp}{}_i$ *acts trivially on V_j, if $i \neq j$,*

(iii) *if $i \geq 1$, $\Phi^{\perp}{}_i$ acts irreducibly on V_i as the isotropy representation of a simple Riemannian symmetric space.*

We sketch the proof. By the Ambrose-Singer holonomy theorem, the normal curvature $R^{\perp}$ and its parallel translates generate the holonomy algebra. Unfortunately $R^{\perp}$ is not a tensor on $\nu_p M$ only. So it does not make sense to apply Simons' construction to it. The idea in [O1] is to define a tensor

$$\mathcal{R}^{\perp} \colon \otimes^3 \nu M \to \nu M$$

which provides the same geometric information as the normal curvature tensor $R^{\perp}$ but has the same algebraic properties as a Riemannian curvature tensor (i.e., it has the same (anti)-symmetry properties and satisfies the first Bianchi identity).

To construct a tensor of type $(1,3)$ on νM we can regard $R^{\perp}$ as a homomorphism $R^{\perp} \colon \bigwedge^2 T_p M \to \bigwedge^2 \nu_p M$ (where $\bigwedge^2$ is the second exterior power), thus $R^{\perp}$ composed with its adjoint operator $R^{\perp *}$ gives rise to an endomorphism $\mathcal{R}^{\perp} \colon \bigwedge^2 \nu_p M \to \bigwedge^2 \nu_p M$, which can be identified with a $(3,1)$ tensor. By the Ricci equations $\langle R^{\perp}_{xy}\xi, \eta \rangle = \langle [A_\xi, A_\eta] x, y \rangle$, $(x, y \in T_p M$, $\xi, \eta \in \nu_p M)$; thus $R^{\perp *}(\xi \wedge \eta) = [A_\xi, A_\eta]$. Hence, for any $\xi_1, \dots, \xi_4 \in \nu_p M$, we have:

$$\langle \mathcal{R}^{\perp}(\xi_1, \xi_2)\xi_3, \xi_4 \rangle = \langle R^{\perp *}(\xi_1 \wedge \xi_2), R^{\perp *}(\xi_3 \wedge \xi_4) \rangle = -\operatorname{tr}([A_{\xi_1}, A_{\xi_2}][A_{\xi_3}, A_{\xi_4}]),$$

since the inner product on $\bigwedge^2$ is given by $\langle A, B \rangle = -\operatorname{tr}(AB)$. From the above formula, one can see that $\mathcal{R}^{\perp}$ is an algebraic curvature tensor; moreover

$\mathcal{R}^\perp$ and its parallel translates still generate the Lie algebra $L(\Phi^{\perp *})$ of $\Phi^{\perp *}$. Note that the scalar curvature of $\mathcal{R}^\perp$ is non positive and vanishes if and only if $\mathcal{R}^\perp$ vanishes. The remaining part of the proof follows some ideas of Cartan and Simons [Sim]. As a first step, using the first Bianchi identity, one can show that, if the action of $\Phi^{\perp *}$ is reducible, also the group $\Phi^{\perp *}$ splits as a product in such a way that (i) and (ii) in the theorem hold. Thus one can concentrate on irreducible holonomy actions. Since a connected Lie subgroup of the orthogonal group acting irreducibly on a vector space is compact [KN, Appendix 5], one gets that $\Phi^{\perp *}$ is compact. To show that a non trivial irreducible normal holonomy action is an s-representation one has to prove that there exists a new non-zero algebraic curvature tensor $\widetilde{\mathcal{R}}^\perp$ which is $\Phi^{\perp *}$-invariant, i.e., it satisfies $g \cdot \widetilde{\mathcal{R}}^\perp = \widetilde{\mathcal{R}}^\perp$, for any $g \in \Phi^{\perp *}$. This is due to the fact that the scalar curvature of $\mathcal{R}^\perp$ is not zero, so also $\mathcal{R}^\perp$ is not zero and one can average it by means of the action of the compact group $\Phi^{\perp *}$ getting a non zero tensor

$$\widetilde{\mathcal{R}}^\perp := \int_{\Phi^{\perp *}} h \cdot \mathcal{R}^\perp , \qquad h \in \Phi^{\perp *} ,$$

since averaging does not change scalar curvature. Clearly $g \cdot \widetilde{\mathcal{R}}^\perp = \widetilde{\mathcal{R}}^\perp$ for any $g \in \Phi^{\perp *}$. Such a $\Phi^{\perp *}$-invariant curvature tensor $\widetilde{\mathcal{R}}^\perp$, by the classical Cartan construction corresponds to an s-representation (cf. the previous section). Since only the curvature tensor is changed, but the holonomy representation is the same, this allows one to say that the irreducible action of the normal holonomy coincides with an s-representation. □

In [HO] E. Heintze and C. Olmos computed the normal holonomy of all s-representations, with the result that all s-representations arise as normal holonomy representations, with eleven exceptions. Up to now, no example is known of a submanifold realizing one of these exceptions as normal holonomy representation. The simplest of these exceptions, since it has rank one, is the isotropy representation of the Cayley projective space represented by $F_4/\mathrm{Spin}(9)$. K. Tezlaff [Te] gave a negative answer to the question whether this representation is the normal holonomy representation of one of the focal manifolds of the inhomogeneous isoparametric hypersurfaces in spheres of Ferus, Karcher and Münzner [FKM], which would be good candidates.

A still open conjecture is that if M is an irreducible and full homogeneous submanifold of the sphere which is not an orbit of an s-representation then the normal holonomy group acts transitively on the unit sphere of the normal space [O3].

The Normal Holonomy Theorem is an important tool in the study of the geometry of submanifolds. We now review some of its important applications. Even though many constructions can be done for submanifolds of space forms, in the sequel we will restrict to submanifolds of Euclidean space. Clearly these results also hold for submanifolds of the sphere, since one can regard them as submanifolds of Euclidean space, but in general not for submanifolds of real hyperbolic space. In the latter case one has a different behaviour in connection with normal holonomy (see [W], [DO]). We will mention some of these peculiarities in the sequel.

Focal manifolds

We begin by recalling the notion of a focal point. Let $E \colon \nu M \to \mathbf{R}^n$ be the map sending $\xi_x \in \nu_x M$ to $x+\xi_x$. A focal point is a critical value $x+\xi_x$ of E. Since the differential of E at ξ_x has the same rank as the matrix $(\mathrm{id}-A_{\xi_x})$, a point $x+\xi_x$ is focal if and only if $\ker(\mathrm{id}-A_{\xi_x}) \neq 0$. If ξ is a parallel normal field and $\dim\ker(\mathrm{id}-A_{\xi_x})$ does not depend on x, then the offset

$$M_\xi := \{x+\xi_x \mid x \in M\}$$

is an immersed submanifold, which is said to be *parallel* to M, if $\ker(\mathrm{id}-A_{\xi_x})$ is trivial (so that no point of M_ξ is focal and M and M_ξ have the same dimension) and *focal*, if $\ker(\mathrm{id}-A_{\xi_x})$ is not trivial.

An important case when $\dim\ker(\mathrm{id}-A_{\xi_x})$ is independent of x occurs if the parallel normal field ξ is in addition *isoparametric*, i.e., A_ξ has constant eigenvalues. In this case, if we diagonalize A_ξ, (letting $\lambda_1,\dots,\lambda_g$ be the different eigenvalues) the corresponding eigendistributions $E_1,\dots,E_g$ are integrable with totally geodesic leaves. We shall denote by $S_i(q)$ the leaf of E_i through $q \in M$.

If $\lambda_i \neq 0$, set $\xi_i := \frac{1}{\lambda_i}\xi$. Then, since $E_i = \ker(\mathrm{id}-A_{\xi_i})$, the focal manifold M_{ξ_i} has tangent space at x given by $\sum_{j\neq i} E_j(x)$ and one says that the eigendistribution E_i is focalized. The submersion $\pi_i \colon M \to M_{\xi_i}$ has $S_i(q)$ as leaf through $\bar{q} = \pi_i(q) = q+\xi_i(q)$. $S_i(q)$ is a totally geodesic submanifold of the affine space $q+\nu_q M \oplus E_i(q)$, which can be identified with the normal space at $\bar{q}$ of the focal manifold M_{ξ_i}. Observe that $-\xi_i(\bar{q})$ belongs to $S_i(q)$ and, if we take the orbit of $-\xi_i(\bar{q})$ under the restricted normal holonomy group ${\Phi^\perp}^*_{M_{\xi_i}}$ of M_{ξ_i}, then ${\Phi^\perp}^*_{M_{\xi_i}} \cdot (-\xi_i(\bar{q}))$ is locally contained in $S_i(q)$ [CO]. An important consequence of the Normal Holonomy Theorem is that if equality holds (even locally) for any index i then M is a submanifold with constant principal curvatures. More precisely

THEOREM 3.3 ([CO]). *Let M be a submanifold of $\mathbf{R}^n$. Let ξ be a parallel isoparametric normal field on M with non zero eigenvalues $\lambda_1, \dots, \lambda_g$ and $\xi_i = \lambda_i^{-1}\xi$. Assume furthermore that, for any i, $S_i(q)$ locally coincides with the orbit ${\Phi^{\perp}}^*_{M_{\xi_i}} \cdot (-\xi_i(\bar{q}))$ of the restricted normal holonomy group of M_{ξ_i} at $\bar{q}$. Then M is a submanifold with constant principal curvatures.*

A crucial remark for the proof is that the assumption of constant principal curvatures is a tensorial property. Indeed, this allows one to check the constancy of the eigenvalues of the shape operator along curves tangent to either vertical or horizontal subspaces (with respect to the submersions $M \to M_{\xi_i}$). Then one uses the fact that the restricted normal holonomy group acts as an s-representation, so that $S_i(q)$ is a totally geodesic submanifold of M with constant principal curvatures.

HOLONOMY TUBES

Another construction which requires normal holonomy is somehow inverse to focalization and consists of the *holonomy tube*.

If $\eta_p \in \nu_p(M)$ the holonomy tube $(M)_{\eta_p}$ at η_p is the image in the exponential map of the normal holonomy subbundle, $\mathrm{Hol}^{\perp}_{\eta_p} M$, that one gets by parallel translating η_p with respect to $\nabla^{\perp}$, along any piecewise differentiable curve in M. More explicitly

$$(M)_{\eta_p} = \{\gamma(1) + \tilde{\eta}_p(1) \mid \gamma\colon [0,1] \to M \text{ is piecewise differentiable, } \gamma(0) = p, \text{ and } \tilde{\eta}_p \text{ is } \nabla^{\perp}\text{-parallel along } \gamma, \text{ with } \tilde{\eta}_p(0) = \eta_p\}.$$

$\mathrm{Hol}^{\perp}_{\eta_p} M$ is always an immersed submanifold of νM and, if the normal holonomy group is compact, in particular if M is simply connected, it is embedded. Most of the time we will need the holonomy tube for local results, so we will assume M to be simply connected. Since the holonomy tube $(M)_{\eta_p}$ is the image in the exponential map of $\mathrm{Hol}^{\perp}_{\eta_p} M$, if 1 is not an eigenvalue of $A_{\tau^{\perp}_{\gamma}\eta_p}$, for any $\nabla^{\perp}$-parallel transport $\tau^{\perp}_{\gamma}\eta_p$ of η_p along any piecewise differentiable curve γ, or, in particular, if $\|\eta_p\|$ is less than the distance between M and the set of its focal points, then the holonomy tube $(M)_{\eta_p}$ is an immersed submanifold of $\mathbf{R}^n$. In this case there is an obvious projection $\pi_{\eta_p}\colon (M)_{\eta_p} \to M$ whose fibres are orbits of the (restricted) normal holonomy group.

An important local property of the holonomy tube is that, if η_p lies on a principal orbit of the restricted normal holonomy group, then the holonomy tube has flat normal bundle [HOT].

Both constructions of parallel (focal) manifolds and holonomy tubes fit together in a general framework of *partial tubes*, which were introduced by S. Carter and A. West [CW].

ISOPARAMETRIC RANK

A useful technique is to combine the two constructions of parallel focal manifolds and that of holonomy tubes. Namely, given a parallel normal isoparametric vector field and a parallel focal manifold M_ξ of M, we pass to a holonomy tube with respect to $-\xi(q)$ (at some q) and then we compare the geometry of M with that of the tube $(M_\xi)_{-\xi(q)}$.

For example, if we do this in the case of the focal manifold which "focalize" an eigendistribution E_i, a restatement of Theorem 3.3 says that if all holonomy tubes $(M_{\xi_i})_{-\xi_i(q)}$ coincide with M locally, then M is a submanifold with constant principal curvatures.

Actually, if ξ is a parallel normal isoparametric field and M is not reducible at any point (i.e., no neighbourhood splits as an extrinsic product), then we have the following.

THEOREM 3.4 ([OW]). *Let M be a submanifold of the Euclidean space and assume that M is not reducible at any point. Let ξ be an isoparametric parallel normal field to M which is not umbilical. Then, if $q \in M$, the holonomy tube $(M_\xi)_{-\xi(q)}$ around the parallel (focal) manifold $M_\xi \subset \mathbf{R}^n$ coincides locally with M.*

As a consequence of Theorem 3.3, we have

THEOREM 3.5 ([CO]). *Let $M \to S^{n-1} \subset \mathbf{R}^n$ be a full submanifold which is not reducible at any point. Suppose that M admits a isoparametric parallel normal field to M which is not umbilical. Then M is a submanifold with constant principal curvatures.*

If one introduces the notion of *isoparametric rank* at q of a submanifold M of Euclidean space as the maximal number of linearly independent parallel isoparametric normal sections (defined in a neighbourhood of q), one can restate the above theorem as a higher rank rigidity result for submanifolds of the Euclidean sphere S^{n-1}.

THEOREM 3.6 ([CO]). *Let $M \to S^{n-1} \subset \mathbf{R}^n$ be a locally irreducible (i.e. it is not reducible at any point) full submanifold with isoparametric rank greater or equal to two. Then M is a submanifold with constant principal curvatures.*

On the other hand, in [OW] it is proved that irreducible and full submanifolds of hyperbolic space must have isoparametric rank zero.

GEOMETRIC CHARACTERIZATION OF SUBMANIFOLDS WITH CONSTANT PRINCIPAL CURVATURES

One can also apply the construction of holonomy tube to give a proof of the geometric characterization of submanifolds with constant principal curvatures (Theorem 3.1). Let M be a submanifold of $\mathbf{R}^n$ and consider, for $\xi_p \in \nu_p M$, the holonomy tube $(M)_{\xi_p}$. Recall that $(M)_{\xi_p}$ has flat normal bundle.

THEOREM 3.7 ([HOT]). *Suppose $\xi_p \in \nu_p M$ lies on a principal orbit of the restricted normal holonomy group and that $\|\xi_p\|$ is less than the focal distance of M. Then $(M)_{\xi_p}$ is isoparametric if and only if M has constant principal curvatures.*

For the proof it is crucial to compare the shape operators A and $\widehat{A}$ of M and $(M)_{\xi_p}$ respectively. In a common normal direction ζ to M and $(M)_{\xi_p}$ one has the "tube formula"

$$A_{\zeta_p} = \widehat{A}_{\zeta_p}|_{\mathcal{H}}\,[(\mathrm{id} - \widehat{A}_{-\xi_p})|_{\mathcal{H}}]^{-1},$$

where $\mathcal{H}$ denotes the horizontal distribution of the submersion $(M)_{\xi_p} \to M$.

As a consequence of Theorem 3.7 one gets Theorem 3.1, i.e., a submanifold M of Euclidean space has constant principal curvatures if and only if it is either isoparametric or a focal manifold of an isoparametric submanifold.

THE HOMOGENEOUS SLICE THEOREM

We have seen that M has constant principal curvatures if all fibres of the projection of a submanifold M onto a full focal manifold M_{ξ_i} (which focalizes an eigendistribution E_i of a parallel isoparametric normal vector field ξ) are homogeneous under the normal holonomy. We now see that the converse is also true as a consequence of the following property of the normal holonomy of a submanifold with constant principal curvatures.

LEMMA 3.1 (HOLONOMY LEMMA [CO]). *Let M be a full submanifold of $\mathbf{R}^n$ with constant principal curvatures. For any $q \in M$, the eigenvalues of the shape operator A locally distinguish different orbits of the restricted normal holonomy group ${\Phi^{\perp}}^*_q$.*

In other terms, if ζ and η belong to different orbits of the normal holonomy group at q then A_ζ and A_η have different eigenvalues.

If M' is an irreducible full isoparametric submanifold and $\pi\colon M' \to M$ is a projection of M' on a focal manifold, a fibre F of π is a union of orbits of the normal holonomy of the focal manifold. The eigenvalues of the shape operator of M on the whole fibre F are constant. Hence, by the Holonomy Lemma, its connected component should consist of only one orbit. Thus, by the Normal Holonomy Theorem, we get the following important result.

THEOREM 3.8 (HOMOGENEOUS SLICE THEOREM [HOT]). *The fibres of the projection of an isoparametric submanifold on a full focal manifold are orbits of an s-representation.*

THE THEOREM OF THORBERGSSON

An isometric action of a Lie group G on $\mathbf{R}^n$ is called *polar* if there is a linear subspace $\Sigma \subset \mathbf{R}^n$ (called a *section*) which meets every G-orbit orthogonally. The easiest example of polar representation is given by the standard action of $\mathrm{SO}(n)$ on $\mathbf{R}^n$. Note that this action is the isotropy representation of the sphere $S^n = \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$. Indeed, a very important class of polar representations is provided by the isotropy representations of symmetric spaces, the s-representations. A special case is given by the adjoint representations of compact Lie groups; in this case, using Lie algebra theory, it is easy to see that any Cartan subalgebra provides a section for the action.

Actually, J. Dadok [Da] classified polar representations on $\mathbf{R}^n$, showing that they are all orbit equivalent to isotropy representations of symmetric spaces (i.e., there is an s-representation with the same orbits).

We have already mentioned that principal orbits of s-representations provide examples of isoparametric submanifolds of Euclidean space. Moreover, if an isoparametric submanifold M is homogeneous, the orbit of a Lie group G acting on $\mathbf{R}^n$ by isometries, then G acts polarly and, by the above mentioned result of J. Dadok, M is an orbit of an s-representation.

The codimension of a homogeneous isoparametric submanifold equals the rank of the symmetric space of the corresponding s-representation. This is one of reasons for which it is customary to call the codimension of an isoparametric submanifold, its *rank*. Clearly another reason is the fact that νM is flat (see later for a more general notion of rank of a submanifold).

In the early 30's, B. Segre showed that the isoparametric hypersurfaces in Euclidean space are parallel hyperplanes, concentric hyperspheres and coaxial cylinders. In particular, all of them are homogeneous. The full and irreducible isoparametric submanifolds of codimension two in Euclidean space, or equivalently, the isoparametric hypersurfaces in spheres, were studied by E. Cartan, who proved that in some cases they are homogeneous, but recognized that this was a much harder object of study. H. Ozeki and M. Takeuchi [OT1] [OT2] in 1975 were the first to find explicit non homogeneous examples and a more systematic approach to find non homogeneous examples was given by D. Ferus, H. Karcher and H. F. Münzner [FKM].

As to higher rank, in 1991, G. Thorbergsson proved the following

THEOREM 3.9 (G. Thorbergsson [Th]). *Any irreducible full isoparametric submanifold of Euclidean space of rank at least three is homogeneous and actually a principal orbit of an s-representation.*

The proof of Thorbergsson uses Tits' buildings and the Homogeneous Slice Theorem. There is an alternative proof of Thorbergsson's result using the theory of homogeneous structures on submanifolds [O2] and normal holonomy. The idea of the proof is the following. By a result in [OS], if there exists on a submanifold M of $\mathbf{R}^n$ a metric connection (called *canonical connection*) ∇^c such that $\nabla^c\alpha = 0$ (where α is the second fundamental form of M) and $\nabla^c(\nabla - \nabla^c) = 0$ (where ∇ is the Levi-Civita connection on M), then M is an orbit of an s-representation. Given an irreducible full isoparametric submanifold of Euclidean space of codimension at least three, one can simultaneously focalize any two eigendistributions. The corresponding fibres are, by the Homogeneous Slice Theorem, orbits of s-representations. Then the canonical connection ∇^c on M is constructed by gluing together the canonical connections that one has naturally on these fibres. The proof of the compatibility between these connections is based on the relation between the normal holonomy groups of the different focal manifolds. The common eigendistributions of the shape operator of M are parallel with respect to the canonical connection. This implies at once that $\nabla^c \alpha = 0$. To show that $\nabla^c(\nabla - \nabla^c) = 0$ one has to use the geometric fact that the ∇^c-parallel transport along a horizontal curve, with respect to some focalization, equals the $\nabla^\perp$-parallel transport in the focal manifold along the projection of the curve. □

Homogeneous submanifolds with higher rank

The last result shows that the orbits of the s-representations agree, up codimension two, with isoparametric submanifolds and their focal manifolds of the Euclidean space. Then it is natural to look for the geometric reasons which guarantee that a (compact) homogeneous submanifold $M = G{\cdot}p$, $\dim(M) \geq 2$, is an orbit of an s-representation. Recall that if M is isoparametric then G acts polarly and then Dadok's theorem implies that M is an orbit of an s-representation. Unfortunately, there exist orbits which are submanifolds with principal curvatures and such that the corresponding isoparametric submanifold (i.e. the holonomy tube) is not homogeneous (see [FKM]). Then it is natural to study how far the dimension of the flat factor of the normal holonomy group of an orbit forces it to be an s-representation orbit. More precisely, let us say that the *rank* of a submanifold is the maximal number of linearly independent (locally defined) parallel normal vectors fields. The following theorem of C. Olmos illustrates how the rank is related to the fact of being an s-representation.

THEOREM 3.10 ([O3]). *Let $G \cdot p = M$,* $\dim(M) \geq 2$, *be an irreducible full homogeneous submanifold (contained in a sphere) of the Euclidean space with* $\operatorname{rank}(M) \geq 2$. *Then M an orbit of the isotropy representation of a simple symmetric space.*

This theorem can be derived from Theorem 3.6 and Theorem 3.9 together with the observation that for homogeneous submanifolds the rank equals the isoparametric rank. This is a consequence of a result, which we will explain in the next section, stating that parallel transport in the maximal parallel flat part of the normal bundle is given by the group action. Thus a parallel normal vector field is isoparametric.

The following result shows that the rank forces an orbit to be contained in a sphere.

THEOREM 3.11 ([O4]). *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space with* $\operatorname{rank}(M) \geq 1$. *Then* M *is contained in a sphere.*

We summarize all the above facts in the following theorem.

THEOREM 3.12. *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space. Then,*

(i) $\operatorname{rank}(M) \geq 1$ *if and only if* M *is contained in a sphere;*

(ii) $\operatorname{rank}(M) \geq 2$ *if and only if* M *is an orbit of an* s*-representation.*

The next corollary uses the fact that the minimal homogeneous submanifolds of Euclidean spaces must be totally geodesic (see [D]).

COROLLARY 3.1. *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space with parallel mean curvature vector* H. *Then* $H \neq 0$ *and* M *is either minimal in a sphere or it is an orbit of an* s*-representation.*

4. HOMOGENEITY AND HOLONOMY

In this section we briefly relate homogeneity and holonomy. In particular, we are interested in the computation of the holonomy group in homogeneous situations. We shall put special emphasis on the tangent bundle of a homogeneous Riemannian manifolds and the normal bundle of a homogeneous submanifold of Euclidean space. But, in the first part, we will work in the framework of arbitrary homogeneous (pseudo)metric vector bundles with a connection. This is because, in our opinion, the main ideas are better understood in this context. Another reason is that one can prove, without extra efforts, very general results which could have applications to the pseudoriemannian case.

Let $E \xrightarrow{\pi} M$ be a finite dimensional real vector bundle over M with a covariant derivative operator ∇ (also called a connection), which corresponds, as usual, to a connection in the sense of distributions. More precisely, there exists a distribution $\mathcal{H}$ on TE such that

(1) $\mathcal{H} \oplus \mathcal{V} = TE$, where $\mathcal{V}$ is the vertical distribution;

(2) $(\mu_c)_*(\mathcal{H}_q) = \mathcal{H}_{\mu_c(q)}$, for all $c \in \mathbf{R}$, where μ_c is multiplication by c.

Let $\langle\ ,\ \rangle$ be a C^∞ metric on the fibres and let g be a Riemannian metric on M (in fact, $\langle\ ,\ \rangle$ and g need not be positive definite). We assume that there is a Lie group G which acts on E by bundle morphisms, whose induced action on M is by isometries and is transitive. Moreover, we assume that the action on E preserves both the metric on the fibres and the connection. A vector X in the Lie algebra $\mathcal{G}$ of G induces, in a natural way, a Killing vector field $\widetilde{X}$ both on E and M, i.e., if $\xi_p \in E$ (resp. $p \in M$) then $\widetilde{X}(\xi_p) := X \,.\, \xi_p := \frac{d}{dt}|_{t=0} \exp(tX)\xi_p$ (resp. $\widetilde{X}(p) := X \,.\, p := \frac{d}{dt}|_{t=0} \exp(tX)p$), where $\exp(tX)$ is the one parameter subgroup associated with X.

We will always keep in mind, as remarked above, the following two important cases:

(a) $M = G/H$ is a homogeneous Riemannian manifold, where G is a Lie subgroup of the isometry group $I(M)$, $E = TM$ is the tangent bundle and ∇ is the usual Levi-Civita connection.

(b) $M = G \cdot v$, where $v \in \mathbf{R}^n$ and G is a Lie subgroup of the isometry group $I(\mathbf{R}^n)$. Here, $E = \nu(M)$ is the normal bundle endowed with the usual normal connection $\nabla^\perp$.

The bundle E is endowed with the so-called Sasaki (Riemannian) metric $\tilde{g}$. Namely,

(i) $\mathcal{H}$ is perpendicular to the vertical distribution $\mathcal{V}$, defined by the tangent space to the fibres $E_q = \pi^{-1}(q)$.

(ii) The restriction of $\tilde{g}$ to $\mathcal{V}$ coincides with the metric on the fibres.

(iii) π is a Riemannian submersion.

The Sasaki metric may be regarded as follows. A curve $\tilde{c}(t)$ in E may be viewed as a section along the curve $c(t) = \pi(\tilde{c}(t))$. In this way, $\tilde{g}(\tilde{c}\,'(0), \tilde{c}\,'(0)) = \langle \frac{D}{dt}|_0 \tilde{c}(t), \frac{D}{dt}|_0 \tilde{c} \rangle + g(c'(0), c'(0))$.

Observe that G acts by isometries, with respect to the Sasaki metric, on E. As is well known, the fibres E_q, $q \in M$, are totally geodesic submanifolds of E. In fact, if $c(t)$ is a curve in M starting at q, then the parallel transport τ_t^c along $c(t)$ defines an isometry from E_q into $E_{c(t)}$. Let $\gamma(s)$ be a curve in E_q and consider $f(s,t) = \tau_t^c(\gamma(s))$. We have that $\langle \tau_t^c(\gamma'(s)), \tau_t^c(\gamma'(s)) \rangle$ does not depend on t and so,

$$0 = \frac{\partial}{\partial t} \tilde{g}\,(\frac{\partial}{\partial s} f, \frac{\partial}{\partial s} f) = 2\tilde{g}\,(\frac{D}{\partial t}\frac{\partial}{\partial s} f, \frac{\partial}{\partial s} f) = 2\tilde{g}\,(\frac{D}{\partial s}\frac{\partial}{\partial t} f, \frac{\partial}{\partial s} f) = 2\langle A_{\frac{\partial}{\partial t} f} \frac{\partial}{\partial s} f, \frac{\partial}{\partial s} f \rangle\,,$$

where A denotes the shape operator of E_q as a submanifold of E. Then E_q is totally geodesic.

We now describe how the holonomy algebra (i.e., the Lie algebra of the holonomy group of the connection ∇ of the bundle $E \xrightarrow{\pi} M$) is linked with the group G. As we saw above the fibres E_q of the bundle E are totally geodesic. Then the projection on E_q of a Killing field $\widetilde{X}$ of E, induced by some $X \in \mathcal{G}$, gives a Killing field $B_q(X)$ of the fibre E_q. Observe that this projection vanishes at 0_q, so $B_q(X)$ belongs to $\mathfrak{so}(E_q)$, the Lie algebra of $\mathrm{SO}(E_q)$. The Lie algebra spanned by these $B_q(X)$ is included in the Lie algebra of the normalizer $N(\mathrm{Hol}_q)$ of the holonomy group Hol_q in $\mathrm{SO}(E_q)$. This is due to the following geometric reason: for any curve c in M and $g \in G$, $\tau_t^{g.c} = g.\tau_t^c.g^{-1}$, since G preserves the connection (and so $g.\mathrm{Hol}_p.g^{-1} = \mathrm{Hol}_{g.p}$, where Hol denotes the holonomy group of the connection on the bundle E).

Let τ_t^X be the flow on E associated to the horizontal component $[\widetilde{X}]^{\mathcal{H}}$ of the Killing field $\widetilde{X}$ (i.e. if $\xi_p \in E_p$, then $\tau_t^X(\xi_p)$ is the parallel transport of ξ_p along the curve $\exp(sX) \cdot p$ from 0 to t). Let F_t^X be the flow of the Killing field $\widetilde{X}$ on E, i.e., $F_t^X(\xi_p) := \exp(tX)\xi_p$. Then the fact that isometries and parallel transport are geometric objects implies that $\tau_t^X \circ F_s^X = F_s^X \circ \tau_t^X$. Taking into account this identity, one finds that $\phi_t := \tau_{-t}^X \circ F_t^X$ defines a one parameter group of isometries of E with the following properties: (i) $\phi_t(E_q) = E_q$, (ii) $\phi_t|_{E_q}$ belongs to $N(\mathrm{Hol}_q)$, the normalizer in $\mathrm{SO}(E_q)$ of the holonomy group Hol_q and (iii) $\phi_t|_{E_q} = e^{B_q(X)t}$, where $B_q(X)$ is the claimed projection of the Killing field X to E_q (i.e. $B_q(X)\xi_q = [X.\xi_q]^{\mathcal{V}}$, where $[\ ]^{\mathcal{V}}$ denotes vertical projection). Note that (iii) is a simple consequence of the general fact that if two flows F_t^X, F_t^Y commute then $F_t^X \circ F_t^Y = F_t^{X+Y}$.

The following theorem makes precise the above description and establishes, using the transitivity of G on M, the inclusion of the holonomy algebra into the Lie algebra generated by the $B_q(X)$.

THEOREM 4.1 ([OSv]). *The Lie algebra $\mathcal{L}_q$ generated by $\{B_q(X) : X \in \mathcal{G}\}$ contains the Lie algebra of the holonomy group* Hol_q *and is contained in the Lie algebra* $N(\mathrm{Hol}_q)$ *of its normalizer in* $\mathrm{SO}(E_q)$.

Proof. In order to illustrate better the main ideas we will only prove a simplified version of the theorem. The inclusion in the normalizer was observed before. Let L_q denote the Lie group associated to $\mathcal{L}_q$ and let $\xi_q \in E_q$. Let us consider $S_{\xi_q} := G \cdot L_q \cdot \xi_q \subset E$. Note that either $S_{\xi_p} \cap S_{\eta_q} = \varnothing$ or $S_{\xi_p} = S_{\eta_q}$, for all $\eta_p, \xi_q \in E$.

It is standard to show that S_{ξ_q} is a subbundle of E over M (of course not a vector subbundle). Observe that the connected component of the fibre at q

of S_{ξ_q} is $L_q \cdot \xi_q$, since the connected component of the isotropy subgroup G_q is contained in L_q. So, the restrictions $\widetilde{X}_{|S_{\xi_q}}$ and $[\widetilde{X}]^{\mathcal{V}}_{|S_{\xi_q}}$ are both tangent to S_{ξ_q} and hence the horizontal component $[\widetilde{X}]^{\mathcal{H}}_{|S_{\xi_q}}$ is also tangent to S_{ξ_q}. Since G acts transitively on M, $\{[\widetilde{X}]^{\mathcal{H}}(\xi_q) : X \in \mathcal{G}\}$ coincides with the horizontal space $\mathcal{H}_{\xi_q}$ (note that $\pi_*(\widetilde{X}) = X$).

Then $\mathcal{H}_\eta \subset T_\eta S_{\xi_q}$ for all $\eta \in S_{\xi_q}$. This implies that $\mathrm{Hol}^*_q \cdot \xi_q \subset L_q \cdot \xi_q$, where Hol^*_q is the connected component of Hol_q (i.e., the restricted holonomy group). In other words, any orbit of Hol^*_q is contained in an orbit of L_q. To get the inclusion $\mathrm{Hol}^* \subset L_q$ one has to carry out a similar argument but replacing E by the principal bundle over M of orthonormal basis of E. □

APPLICATIONS

- $E = TM$, the tangent bundle: in this case we will show that $B_q(X) = (\nabla \bar{X})_q$, where $\bar{X}(p) = X \cdot p$, $p \in M$ (cf. [N]). Indeed,

$$B_q(X) \cdot \xi = \frac{D}{dt}_{|0} \exp(tX) \cdot \xi = \frac{D}{\partial t}_{|0} \frac{\partial}{\partial s}_{|0} \exp(tX) \cdot \gamma_\xi(s)$$
$$= \frac{D}{\partial s}_{|0} \frac{\partial}{\partial t}_{|0} \exp(tX) \cdot \gamma_\xi(s) = \frac{D}{ds}_{|0} X \cdot \gamma_\xi(s) = \nabla_\xi \bar{X},$$

where γ_ξ is the geodesic of M with initial condition $\xi \in E_p$.

If M is locally irreducible and the scalar curvature is not (identically) zero, then the restricted holonomy group Φ^*_q of M is non exceptional, i.e. it acts on T_qM as an s-representation (see [Sim, p. 229]). Then Φ^*_q coincides with the connected component of its normalizer in $\mathrm{SO}(T_qM)$. So, the Lie algebra of Φ^*_q is algebraically generated by $\{B_q(X) : X \in \mathcal{G}\}$. More generally, if M is not Ricci flat the same conclusion holds due to [K] and is now a consequence of next proposition. But Alekseevsky-Kimel′feld [AK] proved that a homogeneous Riemannian manifold cannot be Ricci flat, unless it is flat (a conceptual proof is due to Heintze and appeared in [BB, p. 553]). Then the holonomy algebra can always be calculated in this way for a locally irreducible M (the so-called Kostant's method). The following result is essentially due to Lichnerowicz and it is a consequence of Berger's list [B1]. Since it is difficult to find in the literature we include a simple proof.

PROPOSITION 4.1. *Let M be a Riemannian manifold which is irreducible at $q \in M$ and let $\mathfrak{g}$ be the Lie algebra of the local holonomy group Φ^{loc}_q at q. Let $\mathfrak{n}$ be the normalizer of $\mathfrak{g}$ in $\mathfrak{so}(T_qM)$. Then $\mathfrak{n}$ contains $\mathfrak{g}$ properly if and only if M is Kähler and Ricci flat near q.*

Proof. Let us endow $\mathfrak{so}(T_qM)$ with the usual scalar product $\langle A,B\rangle = -\operatorname{tr}(A.B)$. Assume that $\mathfrak{n} \neq \mathfrak{g}$. If we decompose orthogonally $\mathfrak{n} = \mathfrak{g}\oplus\mathfrak{k}$, then $\mathfrak{g}$ and $\mathfrak{k}$ are ideals of $\mathfrak{n}$ and so $[\mathfrak{g},\mathfrak{k}] = 0$. Now choose $0 \neq J_q \in \mathfrak{k}$. Then J_q^2 is a symmetric endomorphism which commutes with $\mathfrak{g}$. So, J_q^2 commutes with Φ_q^{loc} and then each eigenspace of J_q^2 defines a parallel distribution near q. Since M is locally irreducible at q we conclude, by de Rham's Decomposition Theorem, that $J_q^2 = -c^2\,\mathrm{id}$. We may assume, by rescaling J_q, that $J_q^2 = -\mathrm{id}$. Extending J_q by parallelism we obtain a parallel almost complex structure J on M. Thus, M is Kähler near q. It is well-known [KN, Proposition 4.5, p. 149, vol. II] that the Ricci curvature Ric_M of a Kähler manifold M satisfies:

$$\mathrm{Ric}_M(X,JY) = \frac{\langle R_{X,Y},J\rangle}{2}$$

If γ is any curve in a small neighbourhood of q joining q to p, and τ_γ is the parallel transport along γ, then

$$\langle R_{X_p,Y_p},J_p\rangle = \langle \tau_\gamma^{-1}R_{X_p,Y_p}\tau_\gamma, J_q\rangle = 0$$

since $J_q \perp \mathfrak{g}$. So, M is Ricci flat near q.

The above two formulas, together with the Ambrose-Singer holonomy theorem also show the converse. □

- $E = \nu(M)$, the normal bundle of a submanifold of $\mathbf{R}^n$. Recall that in this case the non trivial part of the normal holonomy representation is an s-representation. Hence, the semisimple part of the normal holonomy group coincides with the connected component of its own normalizer (in the orthogonal group). If M is an irreducible submanifold which is not a curve, then the group G gives the parallel transport in $\nu_0(M)$ (the maximal parallel and flat subbundle of $\nu(M)$ (see [O3]). So, in this case, the Lie algebra of the normal holonomy group is algebraically generated by $\{B_q(X) : X \in \mathcal{G}\}$. Moreover, we have that $B_q(X)$ can be regarded as the projection to the affine subspace $q+\nu_q(M)$ of the Killing field of $\mathbf{R}^n$ (restricted to this normal space) induced by $X \in \mathcal{G}$. So, the normal holonomy group measures how far G is from acting polarly and how far M is from being a principal orbit (in which case this projection would be trivial from the definition of polarity).

Polar actions on the tangent bundle and symmetry

We recall briefly the general notions of polar and hyperpolar actions on Riemannian manifolds; for more details we refer to [Da, PT2, PT1, HPTT]. Let M be a complete Riemannian manifold and let G be a closed subgroup of the full group of isometries of M. A complete embedded and closed submanifold Σ of M is called a *section* if Σ does intersect any orbit of G in M and is perpendicular to orbits at intersection points. If there exists a section in M then the action of G is called a *polar action*. Observe that from a section we can obtain, by means of the group, sections which contain any given point. An action is called *hyperpolar* if it is polar and a section is in addition flat. Of course in the case of $\mathbf{R}^n$ these two concepts coincide.

Let now M be a complete simply connected Riemannian manifold and let TM be its tangent bundle endowed with the Sasaki metric. We will regard M as the (Riemannian) embedded submanifold of TM which consists of the zero vectors. We have the following characterization of symmetric spaces in terms of polar (or equivalently, hyperpolar) actions on TM. The following result was obtained by J. Eschenburg and the third author when writing the article [EO].

THEOREM 4.2. *Let M be a simply connected complete Riemannian manifold. Then the tangent bundle TM admits a polar action having M as an orbit if and only if M is symmetric.*

Proof. Assume M is irreducible. Let G act polarly on TM and $G{\cdot}0_q = M$. If Σ is a section for this action with $q \in \Sigma$ then $\Sigma \subset T_qM$, since horizontal and vertical distributions are perpendicular with respect to the Sasaki metric. Since Σ meets G-orbits perpendicularly, we have that the horizontal distribution of TM is tangent to the G-orbits. Then the parallel transport of any $v \in T_qM$ belongs to $G \cdot v$. If the codimension of $G \cdot v$ is greater than 1, then the holonomy group does not act transitively on the (unit) sphere of T_qM. Hence M is symmetric by the theorem of Berger [B1, Sim]. If $G \cdot v$ has codimension 1 then M must be two point homogeneous and hence symmetric by [Wa] (for a conceptual proof see [Sz]). If $M = M_1 \times \cdots \times M_k$ is reducible, by projecting Killing vector fields to the factors we obtain a bigger group, let us say $\widetilde{G} = G_1 \times \cdots \times G_k$ and such that G_i acts polarly on M_i.

Let us show the converse. As we noted in Section 2, the transvection group Tr(N) acts transitively on any holonomy bundle. Then the polarity follows from the fact the holonomy representation acts polarly. □

It follows from the above results that an irreducible homogeneous space in which holonomy agrees with isotropy must be symmetric.

5. Lorentzian holonomy and homogeneous submanifolds of H^n

In this section we show how the theory of homogeneous submanifolds of the hyperbolic space H^n can be used to obtain general results about the action of a connected Lie subgroup of $O(n,1)$ on the Lorentzian space $\mathbf{R}^{n,1}$, namely,

Theorem 5.1 ([DO]). *Let G be a connected (not necessarily closed) Lie subgroup of $SO(n,1)$ and assume that the action of G on the Lorentzian space $\mathbf{R}^{n,1}$ is weakly irreducible. Then either G acts transitively on H^n or G acts transitively on a horosphere of hyperbolic space. Moreover, if G acts irreducibly, then $G = SO_0(n,1)$.*

We will explain later the concept of weak irreducibility, and we will also sketch the proof of the above theorem. First, we observe that Theorem 5.1 has an immediate corollary, which provides a purely geometric answer to a question posed in [BI].

Corollary 5.1 (M. Berger [B1], [B2]). *Let M be a Lorentzian manifold of dimension n. If the restricted holonomy group acts irreducibly on TM it coincides with $SO_0(n,1)$. In particular, if M is locally symmetric then it has constant sectional curvature.*

Before giving the ideas of the proof of Theorem 5.1, we recall some basic facts on hyperbolic geometry.

Let $(V, \langle , \rangle)$ be a (real) vector space endowed with a nondegenerate symmetric bilinear form of signature $(n,1)$. It is standard to identify V with the Lorentzian space $\mathbf{R}^{n,1}$ and $Aut(\langle , \rangle) \cong O(n,1)$. It is well known that the hyperbolic space H^n can be identified with a connected component of the set of points $p \in \mathbf{R}^{n,1}$ such that $\langle p,p \rangle = -1$. As in the case of the sphere, the distance $d = d(p,q)$ between two points of H^n can be computed by the equation: $\cosh(d) = -\langle p,q \rangle$. This equation comes from the fact that geodesics have the form $\exp(tv_p) = \cosh(\|v_p\|t)p + \sinh(\|v_p\|t)\frac{v_p}{\|v_p\|}$. Observe that a connected subgroup of $O(n,1)$ acts on H^n by isometries. An affine subspace $q+V$ of $\mathbf{R}^{n,1}$ is called *Euclidean*, *Lorentzian* or *degenerate*, depending on whether the restriction of $\langle , \rangle$ to V is positive definite, indefinite or degenerate. A *horosphere* is a submanifold of the hyperbolic space which is obtained by intersecting H^n with an affine degenerate hyperplane. Thus, a degenerate hyperplane $q+V$ produces a foliation of H^n by parallel horospheres. The infinity $H^n(\infty)$ is the set of equivalence classes of asymptotic geodesics. It is not difficult to see that two geodesics $\exp(t.v_p)$ and $\exp(t.v_{p'})$ are asymptotic if and only if $\frac{v_p}{\|v_p\|} + p = \lambda(\frac{v'_p}{\|v'_p\|} + p')$ for some real number λ. As a consequence, we can identify the infinity $H^n(\infty)$ with the set of degenerate hyperplanes $\{\frac{v_p}{\|v_p\|} + p\}^{\perp}$. In this way a point z at infinity defines a foliation of H^n by parallel horospheres and we say that the horosphere Q is centred at $z \in H^n(\infty)$ if Q is a leaf of that foliation. An action of a subgroup G of $O(n,1)$ is called *weakly irreducible* if it leaves invariant only degenerate subspaces.

A fundamental tool in the proof of Theorem 5.1 is the following result.

THEOREM 5.2 ([DO]). *Let G be a connected (not necessarily closed) Lie subgroup of isometries of the hyperbolic space H^n. Then one of the following assertions holds:*

(i) *G has a fixed point.*

(ii) *G has a unique non trivial totally geodesic orbit (possibly the full space).*

(iii) *All orbits are included in horospheres centred at the same point at infinity.*

The following fact plays an important role in the proof of Theorem 5.2: if a connected (not necessarily closed) Lie subgroup of isometries of the hyperbolic space H^n has a totally geodesic orbit (maybe a fixed point) then no other orbit can be minimal [DO]. A simple consequence of this fact and Theorem 5.2 is the following

THEOREM 5.3 ([DO]). *A minimal (extrinsically) homogeneous submanifold of the hyperbolic space must be totally geodesic.*

The same fact is also true in Euclidean space [D] (see also [O4]). On the other hand, it is well-known that there exist many non totally geodesic minimal (extrinsically) homogeneous submanifolds in spheres [H], [HL]. Also, there exist non totally geodesic minimal (extrinsically) homogeneous submanifolds in non compact symmetric spaces [Br]. It is interesting to note that a subgroup G of isometries of the Euclidean space always has a totally geodesic orbit (possibly a fixed point or the whole space).

A key fact in the proof of Theorem 5.2 is the following observation: if a normal subgroup H of a group G of isometries of H^n has a totally geodesic orbit $H \cdot p$ of positive dimension, then $G \cdot p = H \cdot p$. This is because G permutes H-orbits and then one can use the fact that there is a unique totally geodesic orbit, to conclude that $H \cdot p = G \cdot p$.

The next step in proving Theorem 5.2 is to study separately the following two cases: G is semisimple (of noncompact type) and G is not semisimple. In this last case one first proves the theorem for abelian groups. The above observation, applied to a normal abelian subgroup of G, implies that either G must translate a geodesic or G fixes a point at infinity or G admits a proper totally geodesic invariant submanifold. It follows that a connected Lie subgroup G of $O(n, 1)$ which acts irreducibly on $\mathbf{R}^{n,1}$ must be semisimple.

When G is a semisimple Lie group we use an Iwasawa decomposition $G = NAK$. Then one proves that the proper (solvable) subgroup NA of G has a minimal orbit which is also a G-orbit. For this, choose a fixed point p of the compact group K (which always exists by a well known theorem of Cartan). It is possible to prove that the isotropy subgroup G_p of G at p agrees with K. Then the mean curvature vector H of the orbit $G \cdot p = NA \cdot p$ is invariant by the isotropy subgroup at p and, if it is not equal to zero, the G-orbits through points on normal K-invariant geodesics turn out to be homothetical to the orbit $G \cdot p$. Observe that these orbits are also NA orbits. Finally, one can control the volume element of these orbits in terms of Jacobi fields and prove that there exists a minimal G-orbit which is also a NA-orbit.

Finally, one shows that if G has a fixed point z at infinity then either G has a totally geodesic orbit (possibly G acts transitively) or it has fixed points in H^n or all orbits of G are contained in horospheres centred at the same point z at infinity. This is because, when G has neither fixed points in H^n nor orbits in horospheres, one can construct a codimension one normal subgroup N of G such that all N-orbits are contained in the horosphere foliation defined by z. Then N acts on horospheres by isometries and one uses the fact that N must have a totally geodesic orbit in each horosphere (because each horosphere is an Euclidean space). Finally, it is not hard to show that the union of all these totally geodesic orbits over all horospheres is a totally geodesic G-invariant submanifold of H^n. Now an induction argument, involving the dimension of the Lie group G and the dimension of the corresponding hyperbolic space H^n, completes the proof of Theorem 5.2.

The proof of Theorem 5.1 runs as follows: Assume that G does not act transitively in H^n. Then G-orbits must be contained in horospheres. But if an orbit is a proper submanifold of one horosphere, one can construct a proper totally geodesic G-invariant submanifold as the union of the parallel orbits to totally geodesic orbits of the action of G, restricted to the horosphere. Then one obtains a contradiction because totally geodesic submanifolds are obtained by intersecting the hyperbolic space H^n with Lorentzian subspaces. Thus, G must act transitively on each horosphere.

Finally, if G acts irreducibly then G must act transitively on the hyperbolic space and must be semisimple of noncompact type by a previous observation. Then, showing that the isotropy group at some point agrees with a maximal compact subgroup, the second part of the theorem follows from the theory of Riemannian symmetric spaces of noncompact type [He].

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