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**Artikel:** AN HOMOLOGY 4-SPHERE GROUP WITH NEGATIVE DEFICIENCY  
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We shall give an example of an homology 4-sphere whose group has deficiency  $< 0$ . Thus none of the implications “ $G$  is an homology 3-sphere group”  $\Rightarrow$  “ $G$  is finitely presentable, perfect and  $\text{def}(G) = 0$ ”  $\Rightarrow$  “ $G$  is an homology 4-sphere group”  $\Rightarrow$  “ $G$  is finitely presentable and superperfect” can be reversed.

A similar outcome was known for knots by the late 1970s. (Namely, none of the implications “ $G$  is a 1-knot group”  $\Rightarrow$  “ $G$  is a high dimensional knot group and  $\text{def}(G) = 1$ ”  $\Rightarrow$  “ $G$  is a 2-knot group”  $\Rightarrow$  “ $G$  is a high dimensional knot group” can be reversed [Fo62, Ke65, Fa75]). The issue considered here was raised by Plotnick, who suggested a possible example [Pl82]. (See also [Be02]). We use a related construction, but our example is different, and we do not know whether Plotnick’s candidates indeed have negative deficiency.

The construction starts with a 2-knot  $K: S^2 \rightarrow S^4$  and an homology 4-sphere  $\Sigma$ . Let  $M$  be the closed 4-manifold obtained by surgery on  $K$ , and let  $N = M \sharp \Sigma$ . Let  $G = \pi_1(M)$  and  $H = \pi_1(\Sigma)$ . (Thus  $G$  is the group of the knot  $K$ .) Let  $t \in G$  represent a generator of  $G/G' \cong \mathbf{Z}$ , and let  $h \in H$ . The conjugacy class of  $th^{-1} \in \pi_1(N) \cong G * H$  is represented by an unique isotopy class of embeddings of  $S^1$  in  $N$ . Surgery on such an embedding gives an homology 4-sphere  $P$ , with group  $\pi = \pi_1(P) = (G * H) / \langle\langle th^{-1} \rangle\rangle$ .

Let  $\rho = \langle\langle G' \rangle\rangle_\pi$  be the normal closure of the image of  $G'$  in  $\pi$ . Then  $\pi/\rho \cong H$ , and so  $\pi$  is the semidirect product  $\rho \rtimes H$ . Let  $\Gamma = \mathbf{Z}[H]$  and let  $I = \text{Ker}(\varepsilon: \Gamma \rightarrow \mathbf{Z})$  be the augmentation ideal of  $H$ . Since  $H$  is finitely presentable  $I$  has a resolution  $C_*$  by free left  $\Gamma$ -modules which are finitely generated in degrees  $\leq 2$ . Let  $B = H_1(\pi; \Gamma) \cong \rho/\rho'$ . Then  $B$  is a left  $\Gamma$ -module and there is an exact sequence  $0 \rightarrow B \rightarrow A \rightarrow I \rightarrow 0$ , in which  $A = H_1(\pi, 1; \Gamma)$  is a relative homology group [Cr61]. Evaluating the Jacobian matrix associated to a presentation for  $\pi$  via the natural epimorphism from  $\mathbf{Z}[\pi]$  to  $\Gamma$  gives a presentation matrix for  $A$  as a module (see [Cr61] or [Fo62]). Thus there is an exact sequence  $D_*: \dots \rightarrow \Gamma^m \rightarrow \Gamma^n \rightarrow A \rightarrow 0$ , where  $n - m = \text{def}(\pi)$ . A mapping cone construction leads to an exact sequence of the form  $C_2 \oplus D_1 \rightarrow C_1 \oplus D_0 \rightarrow B \oplus C_0 \rightarrow 0$  and hence to a presentation for  $B$  of the form  $C_2 \oplus D_1 \oplus C_0 \rightarrow C_1 \oplus D_0 \rightarrow B$ .

Now let  $K$  be the 2-twist spin of the trefoil knot, with group  $G = \langle x, s \mid x^3 = 1, sxs^{-1} = x^{-1} \rangle$ , and let  $H$  be the Higman group with presentation  $\langle a, b, c, d \mid bab^{-1} = a^2, cbc^{-1} = b^2, dcd^{-1} = c^2, ada^{-1} = d^2 \rangle$ . Then  $H$  is perfect and  $\text{def}(H) = 0$ , so there is an homology 4-sphere  $\Sigma$  with group  $H$ . Moreover  $H$  has cohomological dimension 2 [DV73], and so

there is a short exact sequence  $0 \rightarrow \Gamma^4 \rightarrow \Gamma^4 \rightarrow I \rightarrow 0$ . Let  $t = s$  and  $h = a$ . Then  $\pi = (G * H) / \langle\langle sa^{-1} \rangle\rangle$  has a presentation of deficiency  $-1$ , and  $B \cong \Gamma / \Gamma(3, a + 1)$ . Since  $B \cong \Gamma \otimes_{\Lambda} (\Lambda / \Lambda(3, a + 1))$ , where  $\Lambda = \mathbf{Z}[a, a^{-1}]$ , there is an exact sequence

$$0 \rightarrow \Gamma \xrightarrow{(3, a+1)} \Gamma^2 \xrightarrow{\begin{pmatrix} a+1 \\ -3 \end{pmatrix}} \Gamma \rightarrow B \rightarrow 0.$$

Suppose that  $\pi$  has deficiency 0. Then  $B$  has deficiency 0 as a left  $\Gamma$ -module, by the general argument above. Hence there is an exact sequence

$$0 \rightarrow L \rightarrow \Gamma^p \rightarrow \Gamma^p \rightarrow B \rightarrow 0.$$

Schanuel's Lemma gives an isomorphism  $\Gamma^{1+p+1} \cong L \oplus \Gamma^{p+2}$ , on comparing these two resolutions of  $B$ . The endomorphism of  $\Gamma^{p+2}$  given by projection onto the second summand is an automorphism, by a theorem of Kaplansky (see page 122 of [Ka69]). Hence  $L = 0$  and so  $B$  has a short free resolution. In particular,  $\text{Tor}_2^{\Gamma}(R, B) = 0$  for any right  $\Gamma$ -module  $R$ . But it is easily verified that if  $\bar{B} \cong \Gamma / (3, a + 1)\Gamma$  is the conjugate right  $\Gamma$ -module then  $\text{Tor}_2^{\Gamma}(\bar{B}, B) \neq 0$ . Thus our assumption was wrong, and  $\text{def}(\pi) = -1 < 0$ .

The group of the 2-twist spin of the trefoil knot is the simplest 2-knot group with deficiency 0 [Fo62]. Levine showed that the group of the sum of  $r$  copies of this knot has deficiency  $1 - r$  [Le78]. If we use this sum in our construction above  $\pi$  now has a presentation of deficiency  $-r$  and  $B \cong (\Gamma / \Gamma(3, a + 1))^r$ , so there is an exact sequence

$$0 \rightarrow \Gamma^r \rightarrow \Gamma^{2r} \rightarrow \Gamma^r \rightarrow B \rightarrow 0.$$

Is  $\text{def}(\pi) = -r$ ?

Is there a *finite* homology 4-sphere group of negative deficiency? Our example above is "very infinite" in the sense that the Higman group  $H$  has no finite quotients, and therefore no finite-dimensional representations over any field [Hi51]. The simplest candidate to consider is perhaps the semidirect product of  $\text{SL}(2, \mathbf{F}_5)$  with the normal subgroup  $\mathbf{F}_5^2$ , and with the natural action of  $\text{SL}(2, \mathbf{F}_5)$  on  $\mathbf{F}_5^2$ . (This semidirect product has a presentation with 3 generators and 5 relations, is superperfect, and has order 3000. I do not know whether it is the group of an homology 4-sphere, nor whether it has deficiency 0.)