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HOLONOMY AND SUBMANIFOLD GEOMETRY

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ABSTRACT. We survey some applications of holonomic methods to the study of submanifold geometry, showing the consequences of some sort of extrinsic version of the de Rham Decomposition Theorem and of Berger's Theorem: the so-called *Normal Holonomy Theorem.* At the same time, from geometric methods in submanifold theory we sketch some very strong applications to the holonomy of Lorentzian manifolds. Moreover we give a conceptual modern proof of a result of Kostant for homogeneous spaces.

1. INTRODUCTION

A connection on a connected Riemannian manifold M can be interpreted as a way of comparing tangent spaces at different points, by means of parallel transport.

The parallel translation depends, in general, on the curve joining two points and this dependence is measured by the *holonomy group*, i.e. the linear group of isometries obtained by parallel transporting along based loops.

Actually holonomy groups can be defined for any connection on a vector bundle. For example, in this note we will be particularly interested in the holonomy group of the normal connection, called *normal holonomy group*.

Holonomy plays an important role in (intrinsic) Riemannian geometry, in the context of special Riemannian metrics, e.g., symmetric, Kähler, hyperkähler and quaternionic Kähler metrics.

The main purpose of this note is to survey the application of holonomic methods to the study of submanifold geometry and vice versa. Namely, we will sketch some very strong applications of geometric methods in submanifold

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theory to the holonomy of Lorentzian manifolds. But we will also be interested in Riemannian holonomy and will give a conceptual modern proof of a result of Kostant for homogeneous spaces.

The survey is organized as follows. In Section 2 we recall some important results on holonomy for a Riemannian manifold. This also allows one to make a comparison with results on normal holonomy, to which Section 3 is devoted. The crucial result in the extrinsic geometry is the Normal Holonomy Theorem (3.2) [O1], which asserts that the non trivial part of normal holonomy acting on the normal space is an s-representation (i.e., isotropy representation of a Riemannian symmetric space). Recall that, by Dadok's Theorem, s-representations are orbit equivalent to *polar actions* and that principal orbits of s-representation are *isoparametric submanifolds*. The Normal Holonomy Theorem is somehow an extrinsic analogue of the de Rham Decomposition Theorem and of Berger's Theorem on Riemannian holonomy. One of its main consequences is the recognition that orbits of s-representations play a similar role, in submanifold geometry, as Riemannian symmetric spaces in intrinsic Riemannian geometry. This is illustrated by high rank theorems (Theorems 3.6, 3.9, 3.10), which have similarities with higher rank results on Riemannian manifolds. The extrinsic notion of rank is linked to maximal flat parallel subbundles of the normal bundle.

In Section 4 we relate homogeneity and holonomy in the general framework of homogeneous (pseudo) Riemannian vector bundles endowed with a connection. The Lie algebra of the holonomy group (holonomy algebra) can be described in terms of projection of Killing vector fields on the homogeneous bundle. As an application to Riemannian manifolds we get Kostant's method for computing the Lie algebra of the holonomy group of a homogeneous Riemannian manifold. Moreover we obtain a local characterization of Kähler and Ricci flat Riemannian manifolds in terms of the normalizer of the Lie algebra of the local holonomy group (Proposition 4.1). For a submanifold M which is an orbit of an orthogonal representation of a Lie group G, normal holonomy measures how much G fails to act polarly and how much M fails to be a principal orbit. Polar actions on the tangent bundle of a simply connected Riemannian manifold M allow to characterize symmetric spaces. This is done in Theorem 4.2, which states that the tangent bundle TM admits a polar action having M as an orbit if and only if M is symmetric.

In Section 5 we show how the theory of homogeneous submanifolds of the hyperbolic space H^n can be used to obtain general results on the action of a connected Lie subgroup of $\mathrm{O}(n, 1)$ on the Lorentzian space $\mathbf{R}^{n,1}$. A consequence is a completely geometric proof, using submanifold geometry, of the fact that the restricted holonomy group of an irreducible Lorentzian manifold is $SO_0(n,1)$, [B1], [B2].

At least a sketch of proof is given for every result mentioned. In some cases we include complete proofs if they are hard to find in the literature.

2. Riemannian holonomy

We first recall some basic facts on holonomy. For more details we refer to [Be, Chapter 10]. Let M be a Riemannian manifold, which we will always assume to be connected. If we fix a point $p \in M$, the parallel transport τ_γ along any loop γ at p determines an isometry of T_pM. The set of all such isometries is a subgroup $\Phi_p(M)$ of the orthogonal group $O(T_p(M))$, called the *holonomy subgroup* of M at p. If q is another point of M, β a path from p to q and τ_β the parallel transport along β, we have $\Phi_q(M) = \tau_\beta\, \Phi_p(M)\, \tau_\beta^{-1}$, so that holonomy groups at different points are conjugated and one speaks of holonomy group of M neglecting the base point. There is a variant of this definition, the *restricted* holonomy group $\Phi_p^*(M)$, obtained by considering only those loops which are homotopically trivial. This group actually behaves more nicely: it is a connected, closed Lie subgroup of $SO(T_pM)$ and is in fact the identity component of $\Phi_p(M)$. It can be regarded as the holonomy group of the universal covering space of M.

Holonomy is closely tied to curvature, which is roughly an infinitesimal measure of holonomy. More precisely, the Ambrose-Singer Holonomy Theorem states that the Lie algebra of the holonomy group is spanned by the curvature operators R_{xy}, $x, y \in T_pM$, together with their parallel translates along loops at p.

In order to describe the importance holonomy plays in intrinsic geometry, we recall an important property of holonomy, the so-called *holonomy principle*: evaluation at p establishes a one-to-one correspondence between parallel tensor fields and tensors invariant under holonomy. The existence of holonomy invariant tensors has strong consequences on the geometry. Here are some examples of this situation.

- For a generic metric $\Phi_p(M) = O(T_pM)$, and there is no invariant tensor besides those generated by the metric (see e.g. [Iw]).
- The existence of an invariant subspace implies that the manifold locally splits (de Rham Decomposition Theorem). Thus one can always restrict attention to irreducible holonomy actions.