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Artikel: ON THE CLASSIFICATION OF CERTAIN PIECEWISE LINEAR AND DIFFERENTIABLE MANIFOLDS IN DIMENSION EIGHT AND AUTOMORPHISMS OF $\sharp_{i=1}^b (S^2 \times S^5)$

Kapitel: 3.6 Links of 3-spheres in $\sharp_{i=1}^b (S^2 \times S^5)$

Autor: Schmitt, Alexander

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3.5 Pontrjagin classes and $\pi_3(\mathrm{SO}(4))$

Vector bundles of rank 4 over S^4 are classified by elements in $\pi_3(\mathrm{SO}(4))$. In our setting, such vector bundles will appear as normal bundles. We recall, therefore, the description of that group and relate it to Pontrjagin classes and self intersection numbers.

First, look at the natural map $\pi_3(\mathrm{SO}(4)) \longrightarrow \pi_3\big(\mathrm{SO}(4)/\mathrm{SO}(3)\big) = \pi_3(S^3)$. This map has a splitting ([32], § 22.6) which induces an isomorphism

$$\pi_3(\mathrm{SO}(4)) = \pi_3(\mathrm{SO}(3)) \oplus \pi_3(S^3)\,.$$

Let α_3 be the generator for $\pi_3(\mathrm{SO}(3)) \cong \mathbf{Z}$ from [32], § 22.3, and $\beta_3 := [\mathrm{id}_{S^3}] \in \pi_3(S^3)$, so that we obtain the isomorphism $\mathbf{Z} \oplus \mathbf{Z} \longrightarrow \pi_3(\mathrm{SO}(4))$, $(k_1, k_2) \longmapsto k_1\alpha_3 + k_2\beta_3$. Finally, the kernel of the map $\pi_3(\mathrm{SO}(4)) \longrightarrow \pi_3(\mathrm{SO})$ to the stable homotopy group is generated by $-\alpha_3 + 2\beta_3$ ([32], § 23.6), whence [23], (20.9), implies

Proposition 3.13. *Let E be the vector bundle over S^4 defined by the element $k_1\alpha_3 + k_2\beta_3 \in \pi_3(\mathrm{SO}(4))$. Then*

$$p_1(E) = \pm(2k_1 + 4k_2)\,.$$

Corollary 3.14. *Let $f\colon S^4 \longrightarrow M$ be a differentiable embedding of S^4 into the differentiable 8-manifold M. Let $E := f^*T_M/T_{S^4}$ be the normal bundle. Then the self intersection number s of $f(S^4)$ in M satisfies*

$$2s \equiv p_1(E) \mod 4\,.$$

Proof. If E is given by the element $k_1\alpha_3 + k_2\beta_3 \in \pi_3(\mathrm{SO}(4))$, then $s = k_2$ ([17], (5.4), p. 72). Since $p_1(E) = \pm(2k_2 + 4k_1)$, the claim follows. □

3.6 Links of 3-spheres in $\#_{i=1}^b(S^2 \times S^5)$

If X is a closed E-manifold of dimension 8 with $w_2(X) = 0$, then $W_2 := \#_{i=1}^b(S^2 \times D^6)$, $b = b_2(X)$, by Lemma 3.5. Thus, W_4 is determined by a framed link of 3-spheres in $\partial W_2 = \#_{i=1}^b(S^2 \times S^5)$. Therefore, we will now classify such links.

So, let $W := \#_{i=1}^b(S^2 \times S^5)$ be a b-fold connected sum. We can choose b disjoint 2-spheres S_i^2, $i = 1, \ldots, b$, embedded in W and representing the natural basis of $H_2(W, \mathbf{Z})$. One checks that the homotopy type of W is given up to dimension 4 by the b-fold wedge product $S^2 \vee \cdots \vee S^2$. Suppose we are given a link of b' three-dimensional spheres, i.e., we are given b' differentiable embeddings $g_i\colon S^3 \longrightarrow W$, $i = 1, \ldots, b'$, with $g_i(S^3) \cap g_j(S^3) = \varnothing$ for $i \neq j$.

By the transversality theorem ([17], IV.(2.4)), one sees that we may assume $S_i^2 \cap g_j(S^3) = \varnothing$ for all i and j.

By Corollary 3.9, the ambient isotopy class of the embedding g_k is determined by the element $\varphi_k := [g_k] \in \pi_3(W_k)$, $W_k := W \setminus \bigcup_{j \neq k} g_j(S^3)$, $k = 1, \dots, b'$. We clearly have (compare [8])

$$\pi_3(W_k) = \pi_3\big(\underbrace{S^2 \vee \cdots \vee S^2}_{b\times} \vee \underbrace{S^3 \vee \cdots \vee S^3}_{(b'-1)\times}\big)\,,$$

so that the Hilton-Milnor theorem yields

$$\pi_3(W_k) = \bigoplus_{i=1}^{b} \pi_3(S^2) \oplus \bigoplus_{1 \leq i < j \leq b} \pi_3(S^3) \oplus \bigoplus_{j \neq k} \pi_3(S^3)\,.$$

Hence, we write φ_k as a tuple of integers:

$$\varphi_k = \big(l_i^k,\ i = 1, \dots, b;\quad l_{ij}^k,\ 1 \leq i < j \leq b;\ \lambda_{kj},\ j \neq k\big)\,.$$

Observe that, for $j \neq k$, φ_k is mapped under the natural homomorphism

$$\pi_3(W_k) \longrightarrow H_3(W_k, \mathbf{Z}) \longrightarrow H_3\big(W \setminus g_j(S^3), \mathbf{Z}\big)\ (\cong \mathbf{Z})$$

to the image of the fundamental class of S^3 under g_{j*}. Thus, λ_{kj} is just the 'usual' linking number of the spheres $g_k(S^3)$ and $g_j(S^3)$ in W (compare [8]).

3.7 LINKS OF 5-SPHERES IN S^8

Let $\mathcal{F}C_b^{\mathrm{PL}(C^\infty)}$ be as before, and let $C_b^{\mathrm{PL}(C^\infty)}$ be the group of isotopy classes of piecewise linear (smooth) embeddings of b disjoint copies of S^5 into S^8. For $b = 1$, these groups are studied in [10], [19], and [20]. A brief summary with references of results in the case $b > 1$ is contained in Section 2.6 of [11]. We will review some of this material below.

PROPOSITION 3.15. *We have* $\mathcal{F}C_1^{C^\infty} \cong \mathcal{F}C_1^{\mathrm{PL}} \cong \mathbf{Z}_2$.

Proof. Since $\pi_5(\mathrm{SO}(3)) \cong \mathbf{Z}_2$, the standard embedding of S^5 into S^8 with its two possible framings provides an injection of $\mathbf{Z}_2$ into $\mathcal{F}C_1^{\mathrm{PL}(C^\infty)}$. By Zeeman's unknotting theorem 3.10, the map $\mathbf{Z}_2 \longrightarrow \mathcal{F}C_1^{\mathrm{PL}}$ is an isomorphism. As remarked in Section 2.6 of [11], $\mathcal{F}C_1^{\mathrm{PL}}$ is isomorphic to $\mathcal{F}\vartheta$, the group of h-cobordism classes of framed submanifolds of S^8 which are homotopy 5-spheres. Moreover, by [10] and [19], there is an exact sequence

$$\cdots \longrightarrow \vartheta^6 \longrightarrow \mathcal{F}C_1^{C^\infty} \longrightarrow \mathcal{F}\vartheta \longrightarrow \vartheta^5 \longrightarrow \cdots\,.$$

As the groups ϑ^5 and ϑ^6 of exotic 5- and 6-spheres are trivial [17], our claim is settled. $\square$