

## 3.7 Links of 5-spheres in $S^8$

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **10.08.2024**

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By the transversality theorem ([17], IV.(2.4)), one sees that we may assume  $S_i^2 \cap g_j(S^3) = \emptyset$  for all  $i$  and  $j$ .

By Corollary 3.9, the ambient isotopy class of the embedding  $g_k$  is determined by the element  $\varphi_k := [g_k] \in \pi_3(W_k)$ ,  $W_k := W \setminus \bigcup_{j \neq k} g_j(S^3)$ ,  $k = 1, \dots, b'$ . We clearly have (compare [8])

$$\pi_3(W_k) = \pi_3\left(\underbrace{S^2 \vee \dots \vee S^2}_{b \times} \vee \underbrace{S^3 \vee \dots \vee S^3}_{(b'-1) \times}\right),$$

so that the Hilton-Milnor theorem yields

$$\pi_3(W_k) = \bigoplus_{i=1}^b \pi_3(S^2) \oplus \bigoplus_{1 \leq i < j \leq b} \pi_3(S^3) \oplus \bigoplus_{j \neq k} \pi_3(S^3).$$

Hence, we write  $\varphi_k$  as a tuple of integers:

$$\varphi_k = (l_i^k, i = 1, \dots, b; l_{ij}^k, 1 \leq i < j \leq b; \lambda_{kj}, j \neq k).$$

Observe that, for  $j \neq k$ ,  $\varphi_k$  is mapped under the natural homomorphism

$$\pi_3(W_k) \longrightarrow H_3(W_k, \mathbf{Z}) \longrightarrow H_3(W \setminus g_j(S^3), \mathbf{Z}) (\cong \mathbf{Z})$$

to the image of the fundamental class of  $S^3$  under  $g_{j*}$ . Thus,  $\lambda_{kj}$  is just the ‘usual’ linking number of the spheres  $g_k(S^3)$  and  $g_j(S^3)$  in  $W$  (compare [8]).

### 3.7 LINKS OF 5-SPHERES IN $S^8$

Let  $\mathcal{FC}_b^{\text{PL}(C^\infty)}$  be as before, and let  $\mathcal{C}_b^{\text{PL}(C^\infty)}$  be the group of isotopy classes of piecewise linear (smooth) embeddings of  $b$  disjoint copies of  $S^5$  into  $S^8$ . For  $b = 1$ , these groups are studied in [10], [19], and [20]. A brief summary with references of results in the case  $b > 1$  is contained in Section 2.6 of [11]. We will review some of this material below.

PROPOSITION 3.15. *We have  $\mathcal{FC}_1^{C^\infty} \cong \mathcal{FC}_1^{\text{PL}} \cong \mathbf{Z}_2$ .*

*Proof.* Since  $\pi_5(\text{SO}(3)) \cong \mathbf{Z}_2$ , the standard embedding of  $S^5$  into  $S^8$  with its two possible framings provides an injection of  $\mathbf{Z}_2$  into  $\mathcal{FC}_1^{\text{PL}(C^\infty)}$ . By Zeeman’s unknotting theorem 3.10, the map  $\mathbf{Z}_2 \longrightarrow \mathcal{FC}_1^{\text{PL}}$  is an isomorphism. As remarked in Section 2.6 of [11],  $\mathcal{FC}_1^{\text{PL}}$  is isomorphic to  $\mathcal{F}\vartheta$ , the group of h-cobordism classes of framed submanifolds of  $S^8$  which are homotopy 5-spheres. Moreover, by [10] and [19], there is an exact sequence

$$\dots \longrightarrow \vartheta^6 \longrightarrow \mathcal{FC}_1^{C^\infty} \longrightarrow \mathcal{F}\vartheta \longrightarrow \vartheta^5 \longrightarrow \dots$$

As the groups  $\vartheta^5$  and  $\vartheta^6$  of exotic 5- and 6-spheres are trivial [17], our claim is settled.  $\square$

Let  $L_b \subset C_b^{C^\infty}$  be the subgroup of those embeddings for which the restriction to each component is isotopic to the standard embedding. As observed in Section 2.6 of [11], Zeeman's unknotting theorem 3.10 implies that  $L_b = C_b^{PL}$ . The following result settles Proposition 2.3:

$$\text{COROLLARY 3.16. } \mathcal{FC}_b^{C^\infty} \cong \mathcal{FC}_b^{PL} \cong L_b \oplus \bigoplus_{i=1}^b \mathbf{Z}_2.$$

For the group  $L_b$ , Theorem 1.3 of [11] provides a fairly explicit description as an extension of abelian groups. For this, consider the  $b$ -fold wedge product  $\bigvee_{i=1}^b S^2$  of 2-spheres together with its inclusion  $i: \bigvee_{i=1}^b S^2 \hookrightarrow \times_{i=1}^b S^2$  into the  $b$ -fold product of 2-spheres. Finally, let  $p_i: \bigvee_{i=1}^b S^2 \rightarrow S^2$  be the projection onto the  $i^{\text{th}}$  factor,  $i = 1, \dots, b$ . Set, for  $m = 1, 2, \dots$ ,

$$\Lambda_{b,j}^m := \text{Ker}(\pi_m(p_j): \pi_m(\bigvee_{i=1}^b S^2) \rightarrow \pi_m(S^2)), \quad j = 1, \dots, b,$$

$$\Lambda_b^m := \bigoplus_{j=1}^b \Lambda_{b,j}^m$$

and

$$\Pi_b^m := \text{Ker}(\pi_m(i): \pi_m(\bigvee_{i=1}^b S^2) \rightarrow \bigoplus_{i=1}^b \pi_m(S^2)),$$

and define

$$w_b^m: \Lambda_b^m \rightarrow \Pi_b^{m+1}$$

on  $\Lambda_{b,j}^m$  by  $w_b^m(\alpha) := [\alpha, \iota_i]$ . Here,  $[\cdot, \cdot]$  stands for the Whitehead product inside the homotopy groups of  $\bigvee_{i=1}^b S^2$  and  $\iota_i: S^2 \hookrightarrow \bigvee_{i=1}^b S^2$  for the inclusion of the  $i^{\text{th}}$  factor,  $i = 1, \dots, b$ . Theorem 1.3 of [11] yields in our situation

**THEOREM 3.17.** *There is an exact sequence of abelian groups*

$$0 \rightarrow \text{Coker}(w_b^6) \rightarrow L_b \rightarrow \text{Ker}(w_b^5) \rightarrow 0.$$

We remark that the formulas of Steer [33] might be used for the explicit computation of Whitehead products and thus for the determination

of  $\text{Coker}(w_b^6)$  and  $\text{Ker}(w_b^5)$ . The free part of  $L_b$ , e.g., can be obtained quite easily. We confine ourselves to prove the following important fact.

COROLLARY 3.18. *The group  $L_b$  has positive rank for  $b \geq 2$ .*

*Proof.* Let  $L_b := \bigoplus_{l \geq 1} L_{b,l}$  be the free graded Lie algebra with  $L_{b,1} := \bigoplus_{i=1}^b \mathbf{Z} \cdot e_i$ . For  $l = 2, 3, \dots$ , let  $e_1^l, \dots, e_{d_l}^l$  be a basis for  $L_{b,l}$  consisting of iterated commutators of the  $e_i$ . By assigning  $\iota_i$  to  $e_i$ , every iterated commutator  $c \in L_{b,l}$  in the  $e_i$  defines an element  $\alpha(c) \in \pi_{l+1}(\bigvee_{i=1}^b S^2)$ .

To settle our claim, it is certainly sufficient to show that  $\text{Coker}(w_b^6)$  has positive rank. Now, by the Hilton-Milnor theorem

$$\Pi_b^7 \cong \bigoplus_{l=3}^7 \bigoplus_{k=1}^{d_{l-1}} \pi_7(S^l) \cdot \alpha(e_k^{l-1}).$$

Note that  $\pi_7(S^l)$  is finite for  $l \notin \{4, 7\}$  (see [32] and [35] for the explicit description of those groups). The Hopf fibration  $S^7 \rightarrow S^4$  [32], on the other hand, yields a decomposition  $\pi_7(S^4) \cong \pi_6(S^3) \oplus \pi_7(S^7) \cong \mathbf{Z}_{12} \oplus \mathbf{Z}$ . Therefore, it will suffice to show that the free part of  $\Lambda_b^6$  is mapped to  $\bigoplus_{j=1}^{d_6} \pi_7(S^7) \cdot \alpha(e_j^6)$ . For  $j = 1, \dots, b$ , we have

$$\Lambda_{b,j}^6 \cong \bigoplus_{i \neq j} \pi_6(S^2) \cdot \iota_i \oplus \bigoplus_{l=3}^6 \bigoplus_{k=1}^{d_{l-1}} \pi_6(S^l) \cdot \alpha(e_k^{l-1}).$$

The group  $\pi_6(S^l)$  is finite for  $l < 6$ , and we obviously have  $[\alpha(e_k^5), \iota_j] = \alpha([e_k^5, e_j])$ . If we expand the commutator  $[e_k^5, e_j]$  in the basis  $e_1^6, \dots, e_{d_6}^6$ , we find an expansion for  $[\alpha(e_k^6), \iota_j]$  in terms of the  $\alpha(e_k^6)$ .  $\square$

COROLLARY 3.19. *The set of  $\text{GL}_b(\mathbf{Z})$ -equivalence classes of elements in  $L_b$  is infinite for  $b \geq 2$ .*

*Proof.* We have seen that the  $\text{GL}_b(\mathbf{Z})$ -set  $L_{b,3}$  is contained in the  $\text{GL}_b(\mathbf{Z})$ -set  $L_b$ . The  $\text{GL}_b(\mathbf{Z})$ -action on  $L_{b,3}$  originates from a homomorphism  $\text{GL}_b(\mathbf{Z}) \rightarrow \text{GL}(L_{b,3}) := \text{Aut}_{\mathbf{Z}}(L_{b,3})$ . In particular, any matrix  $g \in \text{GL}_b(\mathbf{Z})$  preserves the absolute value of the determinant of any  $d_3$  elements in  $L_{b,3}$ . This implies, for instance, that  $a \cdot e_1^3$  and  $b \cdot e_1^3$  cannot lie in the same  $\text{GL}_b(\mathbf{Z})$ -orbit, if  $0 \leq a < b$ .  $\square$