2. Riemannian holonomy

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of the fact that the restricted holonomy group of an irreducible Lorentzian manifold is $SO_0(n,1)$, [B1], [B2].

At least a sketch of proof is given for every result mentioned. In some cases we include complete proofs if they are hard to find in the literature.

2. Riemannian holonomy

We first recall some basic facts on holonomy. For more details we refer to [Be, Chapter 10]. Let M be a Riemannian manifold, which we will always assume to be connected. If we fix a point $p \in M$, the parallel transport τ_γ along any loop γ at p determines an isometry of T_pM. The set of all such isometries is a subgroup $\Phi_p(M)$ of the orthogonal group $O(T_p(M))$, called the *holonomy subgroup* of M at p. If q is another point of M, β a path from p to q and τ_β the parallel transport along β, we have $\Phi_q(M) = \tau_\beta\,\Phi_p(M)\,\tau_\beta^{-1}$, so that holonomy groups at different points are conjugated and one speaks of holonomy group of M neglecting the base point. There is a variant of this definition, the *restricted* holonomy group $\Phi_p^*(M)$, obtained by considering only those loops which are homotopically trivial. This group actually behaves more nicely: it is a connected, closed Lie subgroup of $SO(T_pM)$ and is in fact the identity component of $\Phi_p(M)$. It can be regarded as the holonomy group of the universal covering space of M.

Holonomy is closely tied to curvature, which is roughly an infinitesimal measure of holonomy. More precisely, the Ambrose-Singer Holonomy Theorem states that the Lie algebra of the holonomy group is spanned by the curvature operators R_{xy}, $x, y \in T_pM$, together with their parallel translates along loops at p.

In order to describe the importance holonomy plays in intrinsic geometry, we recall an important property of holonomy, the so-called *holonomy principle*: evaluation at p establishes a one-to-one correspondence between parallel tensor fields and tensors invariant under holonomy. The existence of holonomy invariant tensors has strong consequences on the geometry. Here are some examples of this situation.

- For a generic metric $\Phi_p(M) = O(T_pM)$, and there is no invariant tensor besides those generated by the metric (see e.g. [Iw]).
- The existence of an invariant subspace implies that the manifold locally splits (de Rham Decomposition Theorem). Thus one can always restrict attention to irreducible holonomy actions.

• It is a classical result of Cartan that, if the Riemannian curvature tensor of a Riemannian manifold M is invariant under parallel transport, M is locally symmetric, i.e., at each point p in M there exists an open ball $B_r(p)$ such that the corresponding local geodesic symmetry s_p is an isometry. A Riemannian manifold is called a *symmetric space* if at each point $p \in M$ such a local geodesic symmetry extends to a global isometry $s_p \colon M \to M$. Symmetric spaces play a prominent role in Riemannian geometry and are very tightly connected to holonomy.

Indeed, let M be an irreducible symmetric space, which can be represented as a quotient $M = G/K$, where G is the identity component of the isometry group of M and K is the isotropy subgroup at some point $p \in M$. One can show that the isotropy representation of K on T_pM agrees with the (effective) representation of the restricted holonomy group $\Phi_p^*(M)$ on T_pM. Observe that, by the Ambrose-Singer Holonomy Theorem and the invariance of the curvature tensor by parallel transport, the holonomy algebra is spanned by the curvature operators R_{xy}, $x, y \in T_pM$. Now the curvature operators allow to recover the symmetric space by a classical construction due to E. Cartan. We briefly outline this construction, which can actually be carried out for any algebraic curvature tensor on some vector space V (i.e., a tensor with the same algebraic properties as the curvature tensor, including the first Bianchi identity) which is invariant by the action of a group K (i.e. $k \cdot R = R$, for any $k \in K$). Indeed, one can construct an orthogonal symmetric Lie algebra $\mathfrak{g}$, by setting $\mathfrak{g} := \mathfrak{k} \oplus V$ and defining

$$\begin{aligned} [B, C] &= BC - CB, && B, C \in \mathfrak{k}, \\ [x, y] &= R_{xy}, && x, y \in V, \\ [A, z] &= Az, && A \in \mathfrak{k},\ z \in V. \end{aligned}$$

Passing to Lie groups one locally recovers G/K (globally if G/K is simply connected).

Yet another characterization of symmetric spaces in terms of holonomy is the following. One can define the *transvection group* of a Riemannian manifold N as the group $\mathrm{Tr}(N)$ of isometries of N that preserve any holonomy subbundle $\mathrm{Hol}_v N$, $v \in T_pN$. Recall that $\mathrm{Hol}_v N$ is the subset, which turns out to be a subbundle, of the tangent bundle TN obtained by parallel transport of v along any piecewise differentiable curve starting from p. More concretely, $\mathrm{Tr}(N)$ is the group of all isometries φ such that, for any $p \in N$, there exists a piecewise differentiable curve γ joining p and $\varphi(p)$ such that $\varphi_{*p} \colon T_pN \to T_{\varphi(p)}N$ coincides with the parallel transport along γ.

Now, a symmetric space M can be characterized by the fact that the transvection group acts transitively on any holonomy subbundle. This is to say that, for any $p, q \in M$, for any piecewise differentiable curve γ from p to q, there exists an isometry g such that $g(p) = q$ and $g_{*p}\colon T_pM \to T_qM$ coincides with the parallel transport along γ.

- If the Ricci tensor is parallel, then M is a product of Einstein manifolds (see e.g. [Be]).
- If there is a complex structure J on a Riemannian manifold M which is orthogonal and parallel, then M is a *Kähler* manifold. In this case the holonomy group is contained in the unitary group $\mathrm{U}(T_pM)$.

Thus, the existence of a geometric structure on a Riemannian manifold can be read in terms of the holonomy invariance of a tensor and this in turn implies a reduction of the holonomy group to a proper subgroup of $\mathrm{O}(T_pM)$.

A fundamental result for the restricted holonomy group $\Phi_p^*(M)$ of a Riemannian manifold is Berger's Theorem ([B1], see also [Be], [Sal], [Sim]), which classifies the possible irreducible actions of the restricted holonomy group on the tangent space at any point: $\Phi_p^*(M)$ is either transitive on the unit sphere of T_pM or it acts as the isotropy representation of a symmetric space (which is also called s-representation) and M is in fact locally symmetric. If the action of $\Phi_p^*(M)$ is transitive on the unit sphere of T_pM and is irreducible, then $\Phi_p^*(M)$ is one the following groups: $\mathrm{SO}(n)$, $\mathrm{U}(n/2)$ $(n \geq 4)$, $\mathrm{SU}(n/2)$ $(n \geq 4)$, $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n/4)$ $(n \geq 4)$, $\mathrm{Sp}(n/4)$ $(n \geq 4)$, $\mathrm{Spin}(9)$ $(n = 16)$, $\mathrm{Spin}(7)$ $(n = 8)$ or G_2 $(n = 7)$.

The reduction of $\Phi_p^*(M)$ to any of the above groups corresponds to some geometric structure on M.

3. Normal holonomy

Let M be a submanifold of a space of constant curvature $\overline{M}$. We will denote by α its second fundamental form, by A its shape operator sending each normal vector ξ to the self adjoint endomorphism A_ξ of the tangent space T_pM, with $\langle A_\xi x, y\rangle = \langle \alpha(x,y), \xi\rangle$ $(x, y \in T_pM)$ and by νM the normal bundle, endowed with the normal connection $\nabla^\perp$. We denote by ${\Phi^\perp}_p$ the *normal holonomy* at $p \in M$, i.e. the holonomy at p of the normal connection.

Recall that there are mainly two possible reductions for submanifolds of spaces of constant curvature. If a submanifold M of $\overline{M}$ is contained in a totally geodesic submanifold N of $\overline{M}$, then one can regard it as a submanifold of