3. Normal holonomy

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Now, a symmetric space M can be characterized by the fact that the transvection group acts transitively on any holonomy subbundle. This is to say that, for any $p, q \in M$, for any piecewise differentiable curve γ from p to q, there exists an isometry g such that $g(p) = q$ and $g_{*p}\colon T_pM \to T_qM$ coincides with the parallel transport along γ.

- If the Ricci tensor is parallel, then M is a product of Einstein manifolds (see e.g. [Be]).
- If there is a complex structure J on a Riemannian manifold M which is orthogonal and parallel, then M is a *Kähler* manifold. In this case the holonomy group is contained in the unitary group $\mathrm{U}(T_pM)$.

Thus, the existence of a geometric structure on a Riemannian manifold can be read in terms of the holonomy invariance of a tensor and this in turn implies a reduction of the holonomy group to a proper subgroup of $\mathrm{O}(T_pM)$.

A fundamental result for the restricted holonomy group $\Phi^*_p(M)$ of a Riemannian manifold is Berger's Theorem ([B1], see also [Be], [Sal], [Sim]), which classifies the possible irreducible actions of the restricted holonomy group on the tangent space at any point: $\Phi^*_p(M)$ is either transitive on the unit sphere of T_pM or it acts as the isotropy representation of a symmetric space (which is also called s-representation) and M is in fact locally symmetric. If the action of $\Phi^*_p(M)$ is transitive on the unit sphere of T_pM and is irreducible, then $\Phi^*_p(M)$ is one the following groups: $\mathrm{SO}(n)$, $\mathrm{U}(n/2)$ $(n \geq 4)$, $\mathrm{SU}(n/2)$ $(n \geq 4)$, $\mathrm{Sp}(1) \cdot \mathrm{Sp}(n/4)$ $(n \geq 4)$, $\mathrm{Sp}(n/4)$ $(n \geq 4)$, $\mathrm{Spin}(9)$ $(n = 16)$, $\mathrm{Spin}(7)$ $(n = 8)$ or G_2 $(n = 7)$.

The reduction of $\Phi^*_p(M)$ to any of the above groups corresponds to some geometric structure on M.

3. Normal holonomy

Let M be a submanifold of a space of constant curvature $\overline{M}$. We will denote by α its second fundamental form, by A its shape operator sending each normal vector ξ to the self adjoint endomorphism A_ξ of the tangent space T_pM, with $\langle A_\xi x, y\rangle = \langle \alpha(x,y), \xi\rangle$ $(x, y \in T_pM)$ and by νM the normal bundle, endowed with the normal connection $\nabla^\perp$. We denote by ${\Phi^\perp}_p$ the *normal holonomy* at $p \in M$, i.e. the holonomy at p of the normal connection.

Recall that there are mainly two possible reductions for submanifolds of spaces of constant curvature. If a submanifold M of $\overline{M}$ is contained in a totally geodesic submanifold N of $\overline{M}$, then one can regard it as a submanifold of

N and one speaks of reduction of codimension for M. If M does not admit any reduction of codimension, or equivalently if M is not contained in any totally geodesic submanifold of $\bar{M}$, one calls M a *full* submanifold of $\bar{M}$. Another reduction is extrinsic splitting as a product of submanifolds and if a submanifold does not split extrinsically, it is called *irreducible.*

Like Riemannian holonomy, normal holonomy plays a fundamental role in the geometry of submanifolds of spaces of constant curvature.

We discuss the analogies and the differences with the intrinsic case in relation with the existence of holonomy invariant tensors.

- $\Phi^{\perp}{}_p = \mathrm{O}(\nu_p M)$ means that there are no invariant tensors on $\nu_p M$ besides those generated by the metric, like in the intrinsic case. This is the case of a generic submanifold.

- The existence of an invariant subspace for the normal holonomy $\Phi^{\perp}{}_p$ does not imply in general that the submanifold locally splits (both extrinsically and intrinsically). For example, for a submanifold of Euclidean space contained in a sphere, the line determined by the position vector p is an invariant subspace under normal holonomy (it always belongs to the flat part of νM), but such a submanifold does not necessarily split. However, if M is a complex submanifold of $\mathbb{C}^n$, then one has a version of the de Rham Decomposition Theorem: if $\Phi^{\perp}{}_p$ splits, M locally splits as a product of submanifolds [D2].

- In the extrinsic case several tensors play an analogous role to the Riemannian curvature tensor: the second fundamental form α (or equivalently the shape operator A) and the normal curvature $R^{\perp}$ are perhaps the most important. A problem, in the context of normal holonomy, is that these tensors do not take values in the normal spaces only. For this reason, to find holonomy invariant tensors, one has to construct new ones. An important class of tensors with values in the normal spaces only is given by the so-called *higher order mean curvatures* [St]. The mean curvature of order k in direction ξ, $H_k(\xi)$, is the k-th elementary symmetric function of the eigenvalues of the shape operator A_ξ in direction ξ. So, up to a constant, $H_k(\xi)$ is the sum of the k-th powers of the eigenvalues $\{\lambda_i\}$ of A_ξ, i.e., $\sum_{i_1<\cdots<i_k} \lambda_{i_1} \ldots \lambda_{i_k}$. Observe that $H_1(\xi) = \langle H, \xi \rangle$, where H is the mean curvature vector field. Let $h_k(\xi_1, \ldots, \xi_k)$ be the symmetric tensor on νM defined by polarization of $H_k(\xi)$. Suppose that any h_k (or equivalently any H_k) is invariant by parallel transport. Thus $H_k(\xi(t))$ is constant for any parallel normal vector field $\xi(t)$ along any piecewise differentiable curve. Since the sum of the k-th powers of the eigenvalues up to order $m = \dim M$ generate all symmetric polynomials on $\lambda_1, \ldots, \lambda_m$, the characteristic polynomial of $A_{\xi(t)}$ is constant,

so $A_{\xi(t)}$ has constant eigenvalues. Conversely, it is easy to see that if $A_{\xi(t)}$ has constant eigenvalues, any h_k (or equivalently any H_k) is invariant by normal holonomy. A submanifold with this property is called a *submanifold with constant principal curvatures.* The importance of the above *tensorial* definition of a submanifold with constant principal curvatures is illustrated in the proof of Theorem 3.3. This class of submanifolds can be regarded for many reasons (which we will clarify in this note) as the extrinsic analogue of locally symmetric spaces. A very important example of submanifolds with constant principal curvatures is given by the *orbits of s-representations*, i.e. the orbits of the isotropy representations of Riemannian symmetric spaces on their tangent spaces, which play the same role, in submanifold geometry, as symmetric spaces in Riemannian geometry (as we will illustrate later). Orbits of s-representations are intrinsically real partial flag manifolds and classically are called R-spaces.

An important special class of submanifolds with constant principal curvatures is given by those with flat normal bundle, called *isoparametric submanifolds.* Actually they are somehow "generic" among submanifolds with constant principal curvatures. Indeed E. Heintze, C. Olmos and G. Thorbergsson gave the following complete characterization of the submanifolds of space forms with constant principal curvatures.

THEOREM 3.1 ([HOT]). *Let M be a submanifold of a space form. Then M has constant principal curvatures if and only if it is either isoparametric or a focal manifold of an isoparametric submanifold.*

We will define focal manifolds later in this section.

In analogy with what we did in the intrinsic case, for symmetric spaces, one can give a characterization of submanifolds with constant principal curvatures in terms of normal holonomy. Let $\mathrm{Tr}(\nabla^\perp)$ be the transvection group of the normal holonomy. Then the orbits of s-representations can be characterized by the fact that $\mathrm{Tr}(\nabla^\perp)$ acts transitively on any normal holonomy subbundle. More explicitly, M is an orbit of an s-representation if and only if for any $p, q \in M$ and any curve γ on M joining p and q, there exists an isometry g of Euclidean space, leaving the submanifold M invariant, sending p to q and such that

$$g_{*p|\nu_p M} : \nu_p M \longrightarrow \nu_q M$$

coincides with the $\nabla^\perp$-parallel transport along γ [OS].

One can classify the behaviour of normal holonomy action on $\nu_p M$. The starting point for this is a proof of Berger's Theorem, due to J. Simons [Sim], which is based on some algebraic properties of the curvature tensor, namely its antisymmetry properties and the first Bianchi identity.

By similar methods an analogous result was proved in [O1] for the restricted normal holonomy group. Roughly, the non trivial part of normal holonomy acts on $\nu_p M$ as an s-representation. More precisely:

THEOREM 3.2 (NORMAL HOLONOMY THEOREM). *Let M be a submanifold of a space form. Let $p \in M$ and let $\Phi^{\perp *}$ be the restricted normal holonomy group at p. Then $\Phi^{\perp *}$ is compact, there exists a unique (up to order) orthogonal decomposition $\nu_p M = V_0 \oplus \cdots \oplus V_k$ of the normal space at p into $\Phi^{\perp *}$-invariant subspaces and there exist normal subgroups of $\Phi^{\perp *}$, ${\Phi^\perp}_0 = \{1\}, {\Phi^\perp}_1, \ldots, {\Phi^\perp}_k$ such that*

(i) $\Phi^{\perp *} = \{1\} \times {\Phi^\perp}_1 \times \cdots \times {\Phi^\perp}_k$ *(direct product),*

(ii) ${\Phi^\perp}_i$ *acts trivially on V_j, if $i \neq j$,*

(iii) *if $i \geq 1$, ${\Phi^\perp}_i$ acts irreducibly on V_i as the isotropy representation of a simple Riemannian symmetric space.*

We sketch the proof. By the Ambrose-Singer holonomy theorem, the normal curvature $R^\perp$ and its parallel translates generate the holonomy algebra. Unfortunately $R^\perp$ is not a tensor on $\nu_p M$ only. So it does not make sense to apply Simons' construction to it. The idea in [O1] is to define a tensor

$$\mathcal{R}^\perp \colon \otimes^3 \nu M \to \nu M$$

which provides the same geometric information as the normal curvature tensor $R^\perp$ but has the same algebraic properties as a Riemannian curvature tensor (i.e., it has the same (anti)-symmetry properties and satisfies the first Bianchi identity).

To construct a tensor of type $(1,3)$ on νM we can regard $R^\perp$ as a homomorphism $R^\perp \colon \bigwedge^2 T_p M \to \bigwedge^2 \nu_p M$ (where $\bigwedge^2$ is the second exterior power), thus $R^\perp$ composed with its adjoint operator $R^{\perp *}$ gives rise to an endomorphism $\mathcal{R}^\perp \colon \bigwedge^2 \nu_p M \to \bigwedge^2 \nu_p M$, which can be identified with a $(3,1)$ tensor. By the Ricci equations $\langle R^\perp_{xy}\xi, \eta\rangle = \langle [A_\xi, A_\eta]x, y\rangle$, $(x, y \in T_p M$, $\xi, \eta \in \nu_p M)$; thus $R^{\perp *}(\xi \wedge \eta) = [A_\xi, A_\eta]$. Hence, for any $\xi_1, \ldots, \xi_4 \in \nu_p M$, we have:

$$\langle \mathcal{R}^\perp(\xi_1, \xi_2)\xi_3, \xi_4\rangle = \langle R^{\perp *}(\xi_1 \wedge \xi_2), R^{\perp *}(\xi_3 \wedge \xi_4)\rangle = -\operatorname{tr}([A_{\xi_1}, A_{\xi_2}][A_{\xi_3}, A_{\xi_4}]),$$

since the inner product on $\bigwedge^2$ is given by $\langle A, B\rangle = -\operatorname{tr}(AB)$. From the above formula, one can see that $\mathcal{R}^\perp$ is an algebraic curvature tensor; moreover

$\mathcal{R}^\perp$ and its parallel translates still generate the Lie algebra $L(\Phi^{\perp *})$ of $\Phi^{\perp *}$. Note that the scalar curvature of $\mathcal{R}^\perp$ is non positive and vanishes if and only if $\mathcal{R}^\perp$ vanishes. The remaining part of the proof follows some ideas of Cartan and Simons [Sim]. As a first step, using the first Bianchi identity, one can show that, if the action of $\Phi^{\perp *}$ is reducible, also the group $\Phi^{\perp *}$ splits as a product in such a way that (i) and (ii) in the theorem hold. Thus one can concentrate on irreducible holonomy actions. Since a connected Lie subgroup of the orthogonal group acting irreducibly on a vector space is compact [KN, Appendix 5], one gets that $\Phi^{\perp *}$ is compact. To show that a non trivial irreducible normal holonomy action is an s-representation one has to prove that there exists a new non-zero algebraic curvature tensor $\widetilde{\mathcal{R}}^\perp$ which is $\Phi^{\perp *}$-invariant, i.e., it satisfies $g \cdot \widetilde{\mathcal{R}}^\perp = \widetilde{\mathcal{R}}^\perp$, for any $g \in \Phi^{\perp *}$. This is due to the fact that the scalar curvature of $\mathcal{R}^\perp$ is not zero, so also $\mathcal{R}^\perp$ is not zero and one can average it by means of the action of the compact group $\Phi^{\perp *}$ getting a non zero tensor

$$\widetilde{\mathcal{R}}^\perp := \int_{\Phi^{\perp *}} h \cdot \mathcal{R}^\perp , \qquad h \in \Phi^{\perp *} ,$$

since averaging does not change scalar curvature. Clearly $g \cdot \widetilde{\mathcal{R}}^\perp = \widetilde{\mathcal{R}}^\perp$ for any $g \in \Phi^{\perp *}$. Such a $\Phi^{\perp *}$-invariant curvature tensor $\widetilde{\mathcal{R}}^\perp$, by the classical Cartan construction corresponds to an s-representation (cf. the previous section). Since only the curvature tensor is changed, but the holonomy representation is the same, this allows one to say that the irreducible action of the normal holonomy coincides with an s-representation. □

In [HO] E. Heintze and C. Olmos computed the normal holonomy of all s-representations, with the result that all s-representations arise as normal holonomy representations, with eleven exceptions. Up to now, no example is known of a submanifold realizing one of these exceptions as normal holonomy representation. The simplest of these exceptions, since it has rank one, is the isotropy representation of the Cayley projective space represented by $F_4/\operatorname{Spin}(9)$. K. Tezlaff [Te] gave a negative answer to the question whether this representation is the normal holonomy representation of one of the focal manifolds of the inhomogeneous isoparametric hypersurfaces in spheres of Ferus, Karcher and Münzner [FKM], which would be good candidates.

A still open conjecture is that if M is an irreducible and full homogeneous submanifold of the sphere which is not an orbit of an s-representation then the normal holonomy group acts transitively on the unit sphere of the normal space [O3].

The Normal Holonomy Theorem is an important tool in the study of the geometry of submanifolds. We now review some of its important applications. Even though many constructions can be done for submanifolds of space forms, in the sequel we will restrict to submanifolds of Euclidean space. Clearly these results also hold for submanifolds of the sphere, since one can regard them as submanifolds of Euclidean space, but in general not for submanifolds of real hyperbolic space. In the latter case one has a different behaviour in connection with normal holonomy (see [W], [DO]). We will mention some of these peculiarities in the sequel.

FOCAL MANIFOLDS

We begin by recalling the notion of a focal point. Let $E\colon \nu M \to \mathbf{R}^n$ be the map sending $\xi_x \in \nu_x M$ to $x+\xi_x$. A focal point is a critical value $x+\xi_x$ of E. Since the differential of E at ξ_x has the same rank as the matrix $(\mathrm{id}-A_{\xi_x})$, a point $x+\xi_x$ is focal if and only if $\ker(\mathrm{id}-A_{\xi_x}) \neq 0$. If ξ is a parallel normal field and $\dim\ker(\mathrm{id}-A_{\xi_x})$ does not depend on x, then the offset

$$M_\xi := \{x+\xi_x \mid x \in M\}$$

is an immersed submanifold, which is said to be *parallel* to M, if $\ker(\mathrm{id}-A_{\xi_x})$ is trivial (so that no point of M_ξ is focal and M and M_ξ have the same dimension) and *focal*, if $\ker(\mathrm{id}-A_{\xi_x})$ is not trivial.

An important case when $\dim\ker(\mathrm{id}-A_{\xi_x})$ is independent of x occurs if the parallel normal field ξ is in addition *isoparametric*, i.e., A_ξ has constant eigenvalues. In this case, if we diagonalize A_ξ, (letting $\lambda_1,\dots,\lambda_g$ be the different eigenvalues) the corresponding eigendistributions $E_1,\dots,E_g$ are integrable with totally geodesic leaves. We shall denote by $S_i(q)$ the leaf of E_i through $q \in M$.

If $\lambda_i \neq 0$, set $\xi_i := \frac{1}{\lambda_i}\xi$. Then, since $E_i = \ker(\mathrm{id}-A_{\xi_i})$, the focal manifold M_{ξ_i} has tangent space at x given by $\sum_{j\neq i} E_j(x)$ and one says that the eigendistribution E_i is focalized. The submersion $\pi_i\colon M \to M_{\xi_i}$ has $S_i(q)$ as leaf through $\bar q = \pi_i(q) = q+\xi_i(q)$. $S_i(q)$ is a totally geodesic submanifold of the affine space $q+\nu_q M \oplus E_i(q)$, which can be identified with the normal space at $\bar q$ of the focal manifold M_{ξ_i}. Observe that $-\xi_i(\bar q)$ belongs to $S_i(q)$ and, if we take the orbit of $-\xi_i(\bar q)$ under the restricted normal holonomy group ${\Phi^\perp}^*_{M_{\xi_i}}$ of M_{ξ_i}, then ${\Phi^\perp}^*_{M_{\xi_i}}\cdot(-\xi_i(\bar q))$ is locally contained in $S_i(q)$ [CO]. An important consequence of the Normal Holonomy Theorem is that if equality holds (even locally) for any index i then M is a submanifold with constant principal curvatures. More precisely

THEOREM 3.3 ([CO]). *Let M be a submanifold of $\mathbf{R}^n$. Let ξ be a parallel isoparametric normal field on M with non zero eigenvalues $\lambda_1, \ldots, \lambda_g$ and $\xi_i = \lambda_i^{-1}\xi$. Assume furthermore that, for any i, $S_i(q)$ locally coincides with the orbit ${\Phi^{\perp}}^*_{M_{\xi_i}} \cdot (-\xi_i(\bar{q}))$ of the restricted normal holonomy group of M_{ξ_i} at $\bar{q}$. Then M is a submanifold with constant principal curvatures.*

A crucial remark for the proof is that the assumption of constant principal curvatures is a tensorial property. Indeed, this allows one to check the constancy of the eigenvalues of the shape operator along curves tangent to either vertical or horizontal subspaces (with respect to the submersions $M \to M_{\xi_i}$). Then one uses the fact that the restricted normal holonomy group acts as an s-representation, so that $S_i(q)$ is a totally geodesic submanifold of M with constant principal curvatures.

HOLONOMY TUBES

Another construction which requires normal holonomy is somehow inverse to focalization and consists of the *holonomy tube*.

If $\eta_p \in \nu_p(M)$ the holonomy tube $(M)_{\eta_p}$ at η_p is the image in the exponential map of the normal holonomy subbundle, $\mathrm{Hol}^{\perp}_{\eta_p} M$, that one gets by parallel translating η_p with respect to $\nabla^{\perp}$, along any piecewise differentiable curve in M. More explicitly

$$(M)_{\eta_p} = \{\gamma(1) + \tilde{\eta}_p(1) \mid \gamma\colon [0,1] \to M \text{ is piecewise differentiable, } \gamma(0) = p, \text{ and } \tilde{\eta}_p \text{ is } \nabla^{\perp}\text{-parallel along } \gamma, \text{ with } \tilde{\eta}_p(0) = \eta_p\}.$$

$\mathrm{Hol}^{\perp}_{\eta_p} M$ is always an immersed submanifold of νM and, if the normal holonomy group is compact, in particular if M is simply connected, it is embedded. Most of the time we will need the holonomy tube for local results, so we will assume M to be simply connected. Since the holonomy tube $(M)_{\eta_p}$ is the image in the exponential map of $\mathrm{Hol}^{\perp}_{\eta_p} M$, if 1 is not an eigenvalue of $A_{\tau^{\perp}_{\gamma}\eta_p}$, for any $\nabla^{\perp}$-parallel transport $\tau^{\perp}_{\gamma}\eta_p$ of η_p along any piecewise differentiable curve γ, or, in particular, if $\|\eta_p\|$ is less than the distance between M and the set of its focal points, then the holonomy tube $(M)_{\eta_p}$ is an immersed submanifold of $\mathbf{R}^n$. In this case there is an obvious projection $\pi_{\eta_p}\colon (M)_{\eta_p} \to M$ whose fibres are orbits of the (restricted) normal holonomy group.

An important local property of the holonomy tube is that, if η_p lies on a principal orbit of the restricted normal holonomy group, then the holonomy tube has flat normal bundle [HOT].

Both constructions of parallel (focal) manifolds and holonomy tubes fit together in a general framework of *partial tubes*, which were introduced by S. Carter and A. West [CW].

Isoparametric rank

A useful technique is to combine the two constructions of parallel focal manifolds and that of holonomy tubes. Namely, given a parallel normal isoparametric vector field and a parallel focal manifold M_ξ of M, we pass to a holonomy tube with respect to $-\xi(q)$ (at some q) and then we compare the geometry of M with that of the tube $(M_\xi)_{-\xi(q)}$.

For example, if we do this in the case of the focal manifold which "focalize" an eigendistribution E_i, a restatement of Theorem 3.3 says that if all holonomy tubes $(M_{\xi_i})_{-\xi_i(q)}$ coincide with M locally, then M is a submanifold with constant principal curvatures.

Actually, if ξ is a parallel normal isoparametric field and M is not reducible at any point (i.e., no neighbourhood splits as an extrinsic product), then we have the following.

THEOREM 3.4 ([OW]). *Let M be a submanifold of the Euclidean space and assume that M is not reducible at any point. Let ξ be an isoparametric parallel normal field to M which is not umbilical. Then, if $q \in M$, the holonomy tube $(M_\xi)_{-\xi(q)}$ around the parallel (focal) manifold $M_\xi \subset \mathbf{R}^n$ coincides locally with M.*

As a consequence of Theorem 3.3, we have

THEOREM 3.5 ([CO]). *Let $M \to S^{n-1} \subset \mathbf{R}^n$ be a full submanifold which is not reducible at any point. Suppose that M admits a isoparametric parallel normal field to M which is not umbilical. Then M is a submanifold with constant principal curvatures.*

If one introduces the notion of *isoparametric rank* at q of a submanifold M of Euclidean space as the maximal number of linearly independent parallel isoparametric normal sections (defined in a neighbourhood of q), one can restate the above theorem as a higher rank rigidity result for submanifolds of the Euclidean sphere S^{n-1}.

THEOREM 3.6 ([CO]). *Let $M \to S^{n-1} \subset \mathbf{R}^n$ be a locally irreducible (i.e. it is not reducible at any point) full submanifold with isoparametric rank greater or equal to two. Then M is a submanifold with constant principal curvatures.*

On the other hand, in [OW] it is proved that irreducible and full submanifolds of hyperbolic space must have isoparametric rank zero.

GEOMETRIC CHARACTERIZATION OF SUBMANIFOLDS WITH CONSTANT PRINCIPAL CURVATURES

One can also apply the construction of holonomy tube to give a proof of the geometric characterization of submanifolds with constant principal curvatures (Theorem 3.1). Let M be a submanifold of $\mathbf{R}^n$ and consider, for $\xi_p \in \nu_p M$, the holonomy tube $(M)_{\xi_p}$. Recall that $(M)_{\xi_p}$ has flat normal bundle.

THEOREM 3.7 ([HOT]). *Suppose $\xi_p \in \nu_p M$ lies on a principal orbit of the restricted normal holonomy group and that $\|\xi_p\|$ is less than the focal distance of M. Then $(M)_{\xi_p}$ is isoparametric if and only if M has constant principal curvatures.*

For the proof it is crucial to compare the shape operators A and $\widehat{A}$ of M and $(M)_{\xi_p}$ respectively. In a common normal direction ζ to M and $(M)_{\xi_p}$ one has the "tube formula"

$$A_{\zeta_p} = \widehat{A}_{\zeta_p}|_{\mathcal{H}}\,[(\mathrm{id} - \widehat{A}_{-\xi_p})|_{\mathcal{H}}]^{-1}\,,$$

where $\mathcal{H}$ denotes the horizontal distribution of the submersion $(M)_{\xi_p} \to M$.

As a consequence of Theorem 3.7 one gets Theorem 3.1, i.e., a submanifold M of Euclidean space has constant principal curvatures if and only if it is either isoparametric or a focal manifold of an isoparametric submanifold.

THE HOMOGENEOUS SLICE THEOREM

We have seen that M has constant principal curvatures if all fibres of the projection of a submanifold M onto a full focal manifold M_{ξ_i} (which focalizes an eigendistribution E_i of a parallel isoparametric normal vector field ξ) are homogeneous under the normal holonomy. We now see that the converse is also true as a consequence of the following property of the normal holonomy of a submanifold with constant principal curvatures.

LEMMA 3.1 (HOLONOMY LEMMA [CO]). *Let M be a full submanifold of $\mathbf{R}^n$ with constant principal curvatures. For any $q \in M$, the eigenvalues of the shape operator A locally distinguish different orbits of the restricted normal holonomy group ${\Phi^{\perp}}_q^*$.*

In other terms, if ζ and η belong to different orbits of the normal holonomy group at q then A_ζ and A_η have different eigenvalues.

If M' is an irreducible full isoparametric submanifold and $\pi\colon M' \to M$ is a projection of M' on a focal manifold, a fibre F of π is a union of orbits of the normal holonomy of the focal manifold. The eigenvalues of the shape operator of M on the whole fibre F are constant. Hence, by the Holonomy Lemma, its connected component should consist of only one orbit. Thus, by the Normal Holonomy Theorem, we get the following important result.

THEOREM 3.8 (HOMOGENEOUS SLICE THEOREM [HOT]). *The fibres of the projection of an isoparametric submanifold on a full focal manifold are orbits of an s-representation.*

THE THEOREM OF THORBERGSSON

An isometric action of a Lie group G on $\mathbf{R}^n$ is called *polar* if there is a linear subspace $\Sigma \subset \mathbf{R}^n$ (called a *section*) which meets every G-orbit orthogonally. The easiest example of polar representation is given by the standard action of $\mathrm{SO}(n)$ on $\mathbf{R}^n$. Note that this action is the isotropy representation of the sphere $S^n = \mathrm{SO}(n+1)/\,\mathrm{SO}(n)$. Indeed, a very important class of polar representations is provided by the isotropy representations of symmetric spaces, the s-representations. A special case is given by the adjoint representations of compact Lie groups; in this case, using Lie algebra theory, it is easy to see that any Cartan subalgebra provides a section for the action.

Actually, J. Dadok [Da] classified polar representations on $\mathbf{R}^n$, showing that they are all orbit equivalent to isotropy representations of symmetric spaces (i.e., there is an s-representation with the same orbits).

We have already mentioned that principal orbits of s-representations provide examples of isoparametric submanifolds of Euclidean space. Moreover, if an isoparametric submanifold M is homogeneous, the orbit of a Lie group G acting on $\mathbf{R}^n$ by isometries, then G acts polarly and, by the above mentioned result of J. Dadok, M is an orbit of an s-representation.

The codimension of a homogeneous isoparametric submanifold equals the rank of the symmetric space of the corresponding s-representation. This is one of reasons for which it is customary to call the codimension of an isoparametric submanifold, its *rank*. Clearly another reason is the fact that νM is flat (see later for a more general notion of rank of a submanifold).

In the early 30's, B. Segre showed that the isoparametric hypersurfaces in Euclidean space are parallel hyperplanes, concentric hyperspheres and coaxial cylinders. In particular, all of them are homogeneous. The full and irreducible isoparametric submanifolds of codimension two in Euclidean space, or equivalently, the isoparametric hypersurfaces in spheres, were studied by E. Cartan, who proved that in some cases they are homogeneous, but recognized that this was a much harder object of study. H. Ozeki and M. Takeuchi [OT1] [OT2] in 1975 were the first to find explicit non homogeneous examples and a more systematic approach to find non homogeneous examples was given by D. Ferus, H. Karcher and H.F. Münzner [FKM].

As to higher rank, in 1991, G. Thorbergsson proved the following

THEOREM 3.9 (G. Thorbergsson [Th]). *Any irreducible full isoparametric submanifold of Euclidean space of rank at least three is homogeneous and actually a principal orbit of an s-representation.*

The proof of Thorbergsson uses Tits' buildings and the Homogeneous Slice Theorem. There is an alternative proof of Thorbergsson's result using the theory of homogeneous structures on submanifolds [O2] and normal holonomy. The idea of the proof is the following. By a result in [OS], if there exists on a submanifold M of $\mathbf{R}^n$ a metric connection (called *canonical connection*) ∇^c such that $\nabla^c\alpha = 0$ (where α is the second fundamental form of M) and $\nabla^c(\nabla - \nabla^c) = 0$ (where ∇ is the Levi-Civita connection on M), then M is an orbit of an s-representation. Given an irreducible full isoparametric submanifold of Euclidean space of codimension at least three, one can simultaneously focalize any two eigendistributions. The corresponding fibres are, by the Homogeneous Slice Theorem, orbits of s-representations. Then the canonical connection ∇^c on M is constructed by gluing together the canonical connections that one has naturally on these fibres. The proof of the compatibility between these connections is based on the relation between the normal holonomy groups of the different focal manifolds. The common eigendistributions of the shape operator of M are parallel with respect to the canonical connection. This implies at once that $\nabla^c \alpha = 0$. To show that $\nabla^c(\nabla - \nabla^c) = 0$ one has to use the geometric fact that the ∇^c-parallel transport along a horizontal curve, with respect to some focalization, equals the $\nabla^\perp$-parallel transport in the focal manifold along the projection of the curve. □

HOMOGENEOUS SUBMANIFOLDS WITH HIGHER RANK

The last result shows that the orbits of the s-representations agree, up codimension two, with isoparametric submanifolds and their focal manifolds of the Euclidean space. Then it is natural to look for the geometric reasons which guarantee that a (compact) homogeneous submanifold $M = G{\cdot}p$, $\dim(M) \geq 2$, is an orbit of an s-representation. Recall that if M is isoparametric then G acts polarly and then Dadok's theorem implies that M is an orbit of an s-representation. Unfortunately, there exist orbits which are submanifolds with principal curvatures and such that the corresponding isoparametric submanifold (i.e. the holonomy tube) is not homogeneous (see [FKM]). Then it is natural to study how far the dimension of the flat factor of the normal holonomy group of an orbit forces it to be an s-representation orbit. More precisely, let us say that the *rank* of a submanifold is the maximal number of linearly independent (locally defined) parallel normal vectors fields. The following theorem of C. Olmos illustrates how the rank is related to the fact of being an s-representation.

THEOREM 3.10 ([O3]). *Let $G \cdot p = M$,* $\dim(M) \geq 2$*, be an irreducible full homogeneous submanifold (contained in a sphere) of the Euclidean space with* $\operatorname{rank}(M) \geq 2$*. Then M an orbit of the isotropy representation of a simple symmetric space.*

This theorem can be derived from Theorem 3.6 and Theorem 3.9 together with the observation that for homogeneous submanifolds the rank equals the isoparametric rank. This is a consequence of a result, which we will explain in the next section, stating that parallel transport in the maximal parallel flat part of the normal bundle is given by the group action. Thus a parallel normal vector field is isoparametric.

The following result shows that the rank forces an orbit to be contained in a sphere.

THEOREM 3.11 ([O4]). *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space with* $\operatorname{rank}(M) \geq 1$. *Then* M *is contained in a sphere.*

We summarize all the above facts in the following theorem.

THEOREM 3.12. *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space. Then,*

(i) $\operatorname{rank}(M) \geq 1$ *if and only if* M *is contained in a sphere;*

(ii) $\operatorname{rank}(M) \geq 2$ *if and only if* M *is an orbit of an* s*-representation.*

The next corollary uses the fact that the minimal homogeneous submanifolds of Euclidean spaces must be totally geodesic (see [D]).

COROLLARY 3.1. *Let* $G \cdot p = M$, $\dim(M) \geq 2$, *be an irreducible and full homogeneous submanifold of the Euclidean space with parallel mean curvature vector* H. *Then* $H \neq 0$ *and* M *is either minimal in a sphere or it is an orbit of an* s*-representation.*

4. HOMOGENEITY AND HOLONOMY

In this section we briefly relate homogeneity and holonomy. In particular, we are interested in the computation of the holonomy group in homogeneous situations. We shall put special emphasis on the tangent bundle of a homogeneous Riemannian manifolds and the normal bundle of a homogeneous submanifold of Euclidean space. But, in the first part, we will work in the framework of arbitrary homogeneous (pseudo)metric vector bundles with a connection. This is because, in our opinion, the main ideas are better understood in this context. Another reason is that one can prove, without extra efforts, very general results which could have applications to the pseudoriemannian case.

Let $E \xrightarrow{\pi} M$ be a finite dimensional real vector bundle over M with a covariant derivative operator ∇ (also called a connection), which corresponds, as usual, to a connection in the sense of distributions. More precisely, there exists a distribution $\mathcal{H}$ on TE such that

(1) $\mathcal{H} \oplus \mathcal{V} = TE$, where $\mathcal{V}$ is the vertical distribution;

(2) $(\mu_c)_*(\mathcal{H}_q) = \mathcal{H}_{\mu_c(q)}$, for all $c \in \mathbf{R}$, where μ_c is multiplication by c.