4. Rings with the strong $1\frac{1}{2}$ - generator property

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small 0-divisors. In fact, there exist examples of classical rings of quotients R whose Picard group is nontrivial (see, e.g. [La_2 : (2.22)]); such rings are of course Prüfer rings with trivial ideal class groups.

We are now in a position to prove Theorem A stated in the Introduction. In the proof below, *R is assumed to be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors.*

Proof of Theorem A (sketch). For any f. g. projective R-module J of rank 1, let $\{J\}$ denote the isomorphism class of J, viewed as an element in the Picard group $\text{Pic}(R)$. For any f. g. projective R-module P, let $[P] \in K_0(R)$, and let $\overline{[P]}$ denote its image in $\widetilde{K}_0(R) = K_0(R)/\mathbf{Z} \cdot [R]$. We define a map α: $\text{Pic}(R) \to \widetilde{K}_0(R)$ by $\alpha\{J\} = \overline{[J]}$. Note that, for any ideals $I, J \subseteq R$ with J invertible, the R-module J is (projective and hence) flat, so we have an R-module isomorphism $I \otimes_R J \cong IJ$. This and the General Steinitz Isomorphism Theorem readily imply that α is a group homomorphism. Using Theorem 3, we see that α is surjective. Finally, the usual exterior algebra argument gives the injectivity of α. Thus, α: $\text{Pic}(R) \to \widetilde{K}_0(R)$ is a group isomorphism, as desired. □

Having completed the proof of Theorem A, we can easily derive the following corollary on the structure and classification of f. g. projective modules over the Prüfer rings in question. Its proof is essentially the same as that in the classical case of Dedekind domains, so we shall omit it.

COROLLARY 4. *Let R be a Prüfer ring with the $1\frac{1}{2}$ generator property having small 0-divisors. Then any f. g. projective R-module P of rank n is isomorphic to $R^{n-1} \oplus I$ where I is an invertible ideal, and the isomorphism class of P is determined by that of I and the rank n. In particular, f. g. projective R-modules satisfy the cancellation law, and f. g. stably free R-modules are free.*

4. RINGS WITH THE STRONG $1\frac{1}{2}$-GENERATOR PROPERTY

In view of the results in the previous sections, it is of interest to find examples of rings satisfying the $1\frac{1}{2}$ generator property. As it turns out, it is actually easier to name some rings that satisfy a stronger property: let us say that a ring R has the *strong $1\frac{1}{2}$ generator property* if any invertible ideal $I \subseteq R$ is generated by two elements, the first of which can be any prescribed regular element of I. The property has been well studied[3]) for domains (see, for example, [Ka$_2$] and [FS]), and it is straightforward to generalize the notion to arbitrary rings as we did. To show that this is a good generalization, we first point out, in Propositions 5 and 6 below, that two known results in the literature extend nicely from domains to general (commutative) rings.

PROPOSITION 5 (cf. Brewer-Klingler [BK]). *A ring R has the strong $1\frac{1}{2}$ generator property if and only if, for any invertible ideals I, J in R, we have an R-isomorphism $R/J \cong I/JI$.*

Proof (sketch). The "if" part follows by taking $J = Ra$, where a is any given regular element in I. (Note that I/Ra is a quotient of I/JI, which would be cyclic if $I/JI \cong R/J$.) For the "only if" part, consider any invertible ideals $I, J \subseteq R$. Since IJ is also invertible, there exists a regular element $a \in IJ$. Using the strong $1\frac{1}{2}$ generator property, we have $I = Ra + Rb$ for some $b \in I$. As in the proof of [BK: Prop. 1], one shows easily that multiplication by b induces a well-defined R-isomorphism from R/J to I/JI. □

PROPOSITION 6 (cf. Heitmann-Levy [HL]). *Any ring R of Krull dimension ≤ 1 has the strong $1\frac{1}{2}$ generator property.*

Proof (sketch). This was proved for Prüfer domains of Krull dimension 1 in [HL]. However, since we defined the (strong) $1\frac{1}{2}$ generator property to be a property on invertible ideals to begin with, the Prüfer assumption on R becomes unnecessary. The proof in [HL: p. 372] for the domain case can easily be adapted to *any* ring with Krull dimension ≤ 1. In fact, the only additional remark needed for the proof is that, if a is a regular element in such a ring R, then R/Ra, and hence any of its factor rings, has Krull dimension 0. This is true since no prime ideal containing Ra can be a minimal prime of R (according to [Ka$_4$: Th. 84]). □

Of course, Proposition 6 generalizes the well known fact that Dedekind rings have the strong $1\frac{1}{2}$ generator property. In the following, we shall record several other conditions on a ring R, each of which will guarantee that R has the strong $1\frac{1}{2}$ generator property. *Any one of these conditions, coupled with the small 0-divisor property on a Prüfer ring, will thus give us enough assumptions to apply Theorem A.*

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[3]) The terminology we used here follows the one originally introduced by Heitmann and Levy [LS], but not that of Brewer and Klingler [BK] or of Fuchs and Salce [FS].

The sufficient conditions we state below for the strong $1\frac{1}{2}$-generator property are both already in the literature. However, the definitions we adopt in this paper are a bit different from those used in earlier work. For this reason, it will be prudent in recalling these known results to give a brief explanation for each.

Generalizing another definition introduced so far for domains (see, e.g. [FS: p. 97]), we say that a ring R has *finite character* if every *regular* element of R lies in at most finitely many maximal ideals of R; that is, for any regular element $a \in R$, R/Ra is a semilocal ring.

PROPOSITION 7 (Gilmer-Heinzer [GH]). *Any ring R of finite character has the strong $1\frac{1}{2}$ generator property.*

In fact, let $I \subseteq R$ be any invertible ideal, and let a be any given regular element in I. By assumption, a lies only in finitely many maximal ideals of R. Thus, by Theorem 3 of [GH], there exists $b \in I$ such that $I = Ra + Rb$.

PROPOSITION 8 (Griffin [Gr]). *Let R be a ring in which every regular ideal is invertible. Then R is a Prüfer ring of finite character (and hence R has the strong $1\frac{1}{2}$ generator property by Proposition 7).*

The rings in questions are, of course, exactly those Prüfer rings R whose regular ideals are f. g. (or equivalently, satisfy the ACC). By Griffin's Theorem 17 in [Gr], any regular element in such a ring R lies in only finitely many prime ideals of R; in particular, R has finite character, and so Proposition 7 applies. Examples of (commutative) rings satisfying the hypothesis of Proposition 8 include: hereditary rings, local rings whose maximal ideals consist of 0-divisors, and classical rings of quotients (e.g. 0-dimensional rings, such as von Neumann regular rings or perfect rings).

5. NON FINITELY GENERATED PROJECTIVE MODULES

In this section, we turn our attention to possibly non f. g. projective modules, and study the structure of such modules over a Prüfer ring R, assuming again that R has small 0-divisors. The goal of the section will be to prove Theorem B stated in the Introduction. We start by proving the first part of that theorem.