

2. Nonamenability for graphs

Objektyp: **Chapter**

Zeitschrift: **L'Enseignement Mathématique**

Band (Jahr): **48 (2002)**

Heft 3-4: **L'ENSEIGNEMENT MATHÉMATIQUE**

PDF erstellt am: **12.07.2024**

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examples of such subgroups are also possible. For instance, if F is a free group of finite rank and $\phi: F \rightarrow F$ is an atoroidal automorphism, then the mapping torus group of ϕ

$$M_\phi = \langle F, t \mid t^{-1}ft = \phi(f) \text{ for all } f \in F \rangle$$

is word-hyperbolic [8, 13]. In this case $M_\phi/F \simeq \mathbf{Z}$ and hence the Schreier graph for M_ϕ relative to F is amenable.

The author is grateful to Laurent Bartholdi, Philip Bowers, Christophe Pittet and Tatiana Smirnova-Nagnibeda for many helpful discussions regarding random walks, to Pierre de la Harpe and Peter Brinkmann for their careful reading of the paper and numerous valuable suggestions and to Paul Schupp for encouragement.

2. NONAMENABILITY FOR GRAPHS

Let X be a connected graph of bounded degree. We define the *spectral radius* $\rho(X)$ of X as

$$\rho(X) := \limsup_{n \rightarrow \infty} \sqrt[n]{p^{(n)}(x, y)}$$

where x, y are two vertices of X and $p^{(n)}(x, y)$ is the probability that an n -step simple random walk starting at x will end up at y . It is well-known that $\rho(X) \leq 1$ and that the definition of $\rho(X)$ does not depend on the choice of x, y .

DEFINITION 2.1 (Amenability for graphs). A connected graph X of bounded degree is said to be *amenable* if $\rho(X) = 1$ and *nonamenable* if $\rho(X) < 1$.

It is also well-known that nonamenability of X implies that X is *transient*, that is that for a simple random walk on X the probability of ever returning to the basepoint is less than 1 (see for example Theorem 51 of [16]). We refer the reader to [16, 71, 72] for comprehensive background information about random walks on graphs and for further references on this topic.

CONVENTION 2.2. Let X be a connected graph of bounded degree with the simplicial metric d . For a finite nonempty subset $S \subset VX$ we will denote by $|S|$ the number of elements in S .

If S is a finite subset of the vertex set of X and $k \geq 1$ is an integer, we will denote by $\mathcal{N}_k^X(S) = \mathcal{N}_k(S)$ the set of all vertices v of X such that $d(v, S) \leq k$. Also, we will denote $\bar{\partial}^X S = \bar{\partial} S := \mathcal{N}_1(S) - S$.

The number

$$\iota(X) := \inf \left\{ \frac{|\bar{\partial} S|}{|S|} \mid S \text{ is a finite nonempty subset of the vertex set of } X \right\}$$

is called the *Cheeger constant* or the *isoperimetric constant* of X .

There are many alternative definitions of nonamenability:

PROPOSITION 2.3. *Let X be a connected graph of bounded degree with the simplicial metric d . Then the following conditions are equivalent:*

1. *The graph X is nonamenable.*
2. (Følner Criterion) *We have $\iota(X) > 0$.*
3. (Gromov's Doubling Condition) *There is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*

$$|\mathcal{N}_k(S)| \geq 2|S|.$$

4. *For any integer $q > 1$ there is some $k \geq 1$ such that for any finite nonempty subset $S \subseteq VX$ we have*

$$|\mathcal{N}_k(S)| \geq q|S|.$$

5. *For some $0 < \sigma < 1$ we have $p^{(n)}(x, y) = o(\sigma^n)$ for any $x, y \in VX$.*
6. *Let $W(X)$ be the pseudogroup of "bounded perturbations of the identity", that is $W(X)$ consists of all bijections ϕ between subsets of VX such that*

$$\sup_{x \in \text{dom}(\phi)} d(x, \phi(x)) < \infty.$$

Then $W(X)$ admits a "paradoxical decomposition", that is there exist nonempty subsets Y_1, Y_2 of VX and $\phi_1: Y_1 \rightarrow VX$, $\phi_2: Y_2 \rightarrow VX$ such that $\phi_1, \phi_2 \in W(X)$, $VX = Y_1 \sqcup Y_2$ and $\phi_1(Y_1) = \phi_2(Y_2) = VX$.

7. ("Grasshopper Criterion") *There exists a map $\phi: VX \rightarrow VX$ such that*

$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$

and such that for any $x \in VX$ we have $|\phi^{-1}(x)| \geq 2$.

8. *There exists a map $\phi: VX \rightarrow VX$ such that*

$$\sup_{x \in VX} d(x, \phi(x)) < \infty$$

and such that for any $x \in VX$ we have $|\phi^{-1}(x)| = 2$.

9. *The bottom of the spectrum for the combinatorial Laplacian operator on X is > 0 (see [21] for the precise definitions).*
10. *We have $H_0^{uf}(X) = 0$ (see [9] for the precise definition of the uniformly finite homology groups H_i^{uf}).*
11. *We have $H_0^{(l_p)}(X) = 0$ for any $1 < p < \infty$ (see [24] for the precise definition of $H_i^{(l_p)}$).*

All of the above statements are well-known, but we will still provide some sample references. The fact that (1), (2), (5) and (6) are equivalent is stated in Theorem 51 of [16]. The fact that (3), (4), (6), (7) and (8) are equivalent follows from Theorem 32 of [16]. The equivalence of (2) and (9) is due to J. Dodziuk [21]. J. Block and S. Weinberger [9] established the equivalence of (2) and (10). Finally, G. Elek [24] proved that (2) is equivalent to (11).

One can characterize amenability of regular graphs in terms of cogrowth.

DEFINITION 2.4. Let X be a connected graph of bounded degree with a base-vertex x_0 . Let $a_n = a_n(X, x_0)$ be the number of reduced edge-paths of length n from x_0 to x_0 . Let $b_n = b_n(X, x_0)$ be the number of all edge-paths of length n from x_0 to x_0 . Set

$$\alpha(X) := \limsup_{n \rightarrow \infty} \sqrt[n]{a_n} \quad \text{and} \quad \beta(X) := \limsup_{n \rightarrow \infty} \sqrt[n]{b_n}.$$

Then we will call $\alpha(X)$ the *cogrowth rate* of X and we will call $\beta(X)$ the *non-reduced cogrowth rate* of X . These definitions are independent of the choice of x_0 .

It is easy to see that for a d -regular connected graph X we have $\alpha(X) \leq d - 1$ and $\beta(X) \leq d$. Moreover, $\rho(X) = \frac{\beta(X)}{d}$. The following result was originally proved by R. Grigorchuk [39] and J. Cohen [19] for the Cayley graphs of finitely generated groups and by L. Bartholdi [5] for arbitrary regular graphs.

THEOREM 2.5 ([5]). *Let X be a connected d -regular graph with $d \geq 3$. Set $\alpha = \alpha(X)$, $\beta = \beta(X)$ and $\rho = \rho(X)$. Then*

$$\rho = \begin{cases} \frac{2\sqrt{d-1}}{d} & \text{if } 1 \leq \alpha \leq \sqrt{d-1} \\ \frac{\sqrt{d-1}}{d} \left(\frac{\sqrt{d-1}}{\alpha} + \frac{\alpha}{\sqrt{d-1}} \right) & \text{if } \sqrt{d-1} \leq \alpha \leq d-1. \end{cases}$$

In particular $\rho < 1 \iff \alpha < d - 1 \iff \beta < d$.