

Introduction

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A TRIPLE RATIO ON THE UNITARY STIEFEL MANIFOLD

by Jean-Louis CLERC

ABSTRACT. For the unitary Stiefel manifold S realized as the Shilov boundary of the unit ball D in $\text{Mat}(p \times q, \mathbf{C})$, we construct characteristic invariants for the (generic) orbits of the conformal group $\mathbf{PSU}(p, q)$ in $S \times S \times S$. The construction uses the automorphy kernel of the bounded symmetric domain.

INTRODUCTION

Let $D = G/K$ be a bounded symmetric domain in a complex vector space \mathbf{C}^N , and let S be its Shilov boundary. The action of G extends to S and this action is transitive on S . It is generally referred to in the literature as the *conformal action* of G on S . One can show that the action is almost 2-transitive in the sense that G has a dense open orbit in $S \times S$. Hence it is a natural question to look for the G -orbits in $S \times S \times S$ and for characteristic invariants of this action. If D happens to be of tube type (in which case $\dim_{\mathbf{R}} S = \dim_{\mathbf{C}} D$), this question was solved in [CØ]. There are a finite number of open orbits in $S \times S \times S$, and the (generalized) *Maslov index* we constructed is a characteristic invariant for the G -action. In the case of the unit ball in \mathbf{C}^2 , the Shilov boundary coincides with the topological boundary, namely the unit sphere $S = \mathbf{S}^3$. In [Ca], E. Cartan constructed a (real-valued) invariant for triples on S (he called S the “hypersphere”). Independently (and more than 50 years later) Korányi and Reimann studied the case of the unit ball in \mathbf{C}^n (see [KR]). Through the Cayley transform, the problem is changed into an equivalent problem for the Heisenberg group \mathbf{H}_n under the action of its conformal group $G = \mathbf{PSU}(n + 1, 1)$. For this situation, they studied a complex cross ratio on \mathbf{H}_n , from which they were able (in a rather indirect way) to construct a (real-valued) invariant for triples, which characterizes the G -orbits of triples in \mathbf{H}_n . Here we solve the problem for the case where D

is the unit ball in the matrix space $\text{Mat}(p \times q, \mathbf{C})$, S is the unitary Stiefel manifold $\mathbf{S}_{p,q}$ and $G = \mathbf{PSU}(p, q)$. The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain D is used to build a kernel for triples of points inside D which transforms nicely under the action of G . It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary S . This is only possible for triples satisfying a generic condition called *transversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action $(g, X) \mapsto gXg^*$ of GL_q on $\text{Mat}(q \times q, \mathbf{C})$ (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

1. GEOMETRIC SETTING

Let p, q be two integers with $1 \leq q \leq p$, and let

$$(1) \quad D = \{z \in \text{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^*z \gg 0\}.$$

Let $G = \mathbf{SU}(p, q) \subset \text{GL}(p + q, \mathbf{C})$. An element $g \in \text{GL}(p + q, \mathbf{C})$ will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \text{Mat}(p \times p, \mathbf{C}), \quad b \in \text{Mat}(p \times q, \mathbf{C}), \quad c \in \text{Mat}(q \times p, \mathbf{C}), \quad d \in \text{Mat}(q \times q, \mathbf{C}).$$

In this notation, the conditions for g to belong to $\mathbf{U}(p, q, \mathbf{C})$ can be written as

$$(2) \quad \begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q. \end{aligned}$$

Define an action of the group $\text{GL}(p + q, \mathbf{C})$ on $\text{Mat}(p \times q, \mathbf{C})$ by

$$(3) \quad g(z) = (az + b)(cz + d)^{-1}.$$