1. Geometric setting

is the unit ball in the matrix space $\mathrm{Mat}(p \times q, \mathbf{C})$, S is the unitary Stiefel manifold $\mathbf{S}_{p,q}$ and $G = \mathbf{PSU}(p,q)$. The invariant we construct for triples is of matrix-valued nature (it is a conjugacy class) and we give two versions of it (see Theorems 4.3 and 4.4). The basic strategy is to approach the Shilov boundary from inside. The (matrix-valued) *automorphy kernel* for the domain D is used to build a kernel for triples of points inside D which transforms nicely under the action of G. It remains to look carefully at the boundary behaviour of the kernel when the points approach the Shilov boundary S. This is only possible for triples satisfying a generic condition called *tranversality* (see Proposition 2.1 for a definition). The *Cayley transform* plays an important role in the proofs. Finally the problem is reduced to a *linear* problem, which is related to the description of some orbits for the action $(g,X) \mapsto gXg^*$ of GL_q on $\mathrm{Mat}(q \times q, \mathbf{C})$ (see Theorem 3.9).

For general references on bounded symmetric domains and their geometric properties, see [S], and Part III in [Fal]. For explicit calculations related to our example, see [P] and [H].

1. Geometric setting

Let p, q be two integers with $1 \leq q \leq p$, and let

(1) $$D = \{z \in \mathrm{Mat}(p \times q, \mathbf{C}) \mid \mathbf{1}_q - z^*z \gg 0\}\,.$$

Let $G = \mathrm{SU}(p,q) \subset \mathrm{GL}(p+q, \mathbf{C})$. An element $g \in \mathrm{GL}(p+q, \mathbf{C})$ will often be written as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where

$$a \in \mathrm{Mat}(p \times p, \mathbf{C}),\ b \in \mathrm{Mat}(p \times q, \mathbf{C}),\ c \in \mathrm{Mat}(q \times p, \mathbf{C}),\ d \in \mathrm{Mat}(q \times q, \mathbf{C})\,.$$

In this notation, the conditions for g to belong to $\mathrm{U}(p,q,\mathbf{C})$ can be written as

(2) $$\begin{aligned} a^*a - c^*c &= \mathbf{1}_p \\ b^*a - d^*c &= 0 \\ d^*d - b^*b &= \mathbf{1}_q\,. \end{aligned}$$

Define an action of the group $\mathrm{GL}(p+q, \mathbf{C})$ on $\mathrm{Mat}(p \times q, \mathbf{C})$ by

(3) $$g(z) = (az+b)(cz+d)^{-1}\,.$$

The action is not everywhere defined, but it is certainly defined if $g \in G$ and $z \in D$. It defines an action of G on D, and G (or rather $\mathbf{PSU}(p,q)$) is the neutral component of the group of all biholomorphic transformations of D.

The stabilizer of the base point $0 \in D$ is the maximal compact subgroup $K = S(\mathrm{U}(p) \times \mathrm{U}(q))$. Its complexification is the complex group $K^{\mathbf{C}} = S(\mathrm{GL}(p,\mathbf{C}) \times \mathrm{GL}(q,\mathbf{C}))$. We also define the following subgroups

$$P^+ = \left\{ \begin{pmatrix} \mathbf{1}_p & z \\ 0 & \mathbf{1}_q \end{pmatrix}, \; z \in \mathrm{Mat}(p \times q, \mathbf{C}) \right\}$$

$$P^- = \left\{ \begin{pmatrix} \mathbf{1}_p & 0 \\ w & \mathbf{1}_q \end{pmatrix}, \; w \in \mathrm{Mat}(q \times p, \mathbf{C}) \right\}.$$

The corresponding *Harish Chandra decomposition* is the following identity

$$(4) \qquad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \mathbf{1}_p & bd^{-1} \\ 0 & \mathbf{1}_q \end{pmatrix} \begin{pmatrix} a - bd^{-1}c & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} \mathbf{1}_p & 0 \\ d^{-1}c & \mathbf{1}_q \end{pmatrix}$$

valid for $g \in \mathrm{GL}(p+q, \mathbf{C})$ if d is invertible.

The *automorphy kernel* $k(z,w)$ is defined for $z, w \in \mathrm{Mat}(p \times q, \mathbf{C})$ wherever it makes sense by the formula

$$(5) \qquad k(z,w) = (\mathbf{1}_q - w^* z)^{-1}.$$

In particular it is always well defined for $z, w \in D$ and has values in $\mathrm{GL}(q, \mathbf{C})$. It has the following law of transformation for $g \in G$

$$(6) \qquad k(g(z), g(w)) = j(g,z)\, k(z,w)\, j(g,w)^*,$$

where

$$(7) \qquad j(g,z) = cz + d.$$

The *Shilov boundary* of D is the unitary Stiefel manifold S defined by

$$(8) \qquad S = \left\{ \sigma \in \mathrm{Mat}(p \times q, \mathbf{C}) \mid \sigma^* \sigma = \mathbf{1}_q \right\}.$$

The action of G extends to S, and it is clearly transitive on S. In fact the action of K is already transitive.

To go further, we need to make a specific choice of a base point in S. For this we first systematically write elements in $\mathrm{Mat}(p \times q, \mathbf{C})$ as

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix}$$

where $z_q \in \mathrm{Mat}(q \times q, \mathbf{C})$ and $z' \in \mathrm{Mat}\left((p-q) \times q, \mathbf{C}\right)$. With this convention, let $ie = \begin{pmatrix} i\mathbf{1}_q \\ 0 \end{pmatrix}$ be the base point in S. Associated to this choice is the Cayley transform c, given by

$$z = \begin{pmatrix} z_q \\ z' \end{pmatrix} \longmapsto c(z) = \begin{pmatrix} w_q \\ w' \end{pmatrix}$$

with

$$\text{(9)} \qquad \begin{aligned} w_q &= (z_q + i\mathbf{1}_q)(iz_q + \mathbf{1}_q)^{-1} \\ w' &= -z'(iz_q + \mathbf{1}_q)^{-1} . \end{aligned}$$

The inverse of the Cayley transform is the map which to $\begin{pmatrix} w_q \\ w' \end{pmatrix}$ associates the matrix $\begin{pmatrix} z_q \\ z' \end{pmatrix}$ given by

$$\text{(10)} \qquad \begin{aligned} z_q &= (iw_q - \mathbf{1}_q)^{-1}(i\mathbf{1}_q - w_q) \\ z' &= 2w'(iw_q - \mathbf{1}_q)^{-1} . \end{aligned}$$

The Cayley transform is a rational map, well defined on D. The image of D is the *Siegel domain of type II* defined by

$$\text{(11)} \qquad {}^cD = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix} , \frac{1}{2i}(w_q - w_q^*) - w'^*w' \gg 0 \right\}$$

and the image of the Shilov boundary (more exactly the part of the Shilov boundary where the Cayley transform is defined) is

$$\text{(12)} \qquad {}^cS = \left\{ \begin{pmatrix} w_q \\ w' \end{pmatrix} , \frac{1}{2i}(w_q - w_q^*) = w'^*w' \right\} .$$

To the data

$$\begin{aligned} &w_0 \in \mathrm{Mat}((p-q) \times q, \mathbf{C}) \\ &h \in \mathrm{GL}(q, \mathbf{C}),\ u \in \mathrm{U}(p-q, \mathbf{C}),\ \text{such that } \det h = (\det u)^{-1} \\ &s \in \mathrm{Herm}(q, \mathbf{C}) \end{aligned}$$

we associate the transform

$$\text{(13)} \qquad \begin{aligned} w_q &\longmapsto h^*w_qh + s + 2iw_0^*uw'h + iw_0^*w_0 \\ w' &\longmapsto uw'h + w_0 . \end{aligned}$$

Any such transform maps ${}^cD$ in a one-to-one fashion into itself. These transforms form a group and it is exactly the group of affine holomorphic transforms of the domain ${}^cD$.

Let B be the stabilizer of the point ie in G. The conjugate group under the Cayley transform is ${}^cB = c \circ B \circ c^{-1}$ and it turns out to be exactly the group of affine transforms of ${}^cD$ we just described. Observe that the group ${}^cB$ is transitive on ${}^cD$ and on ${}^cS$.