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J.-L. CLERC

3. Orbits for the GL_q -action on \widetilde{T}_q

Any $z \in \operatorname{Mat}(q \times q, \mathbb{C})$ can be written in a unique way as z = x + iy, with $x, y \in H_q$. We will be concerned with the set \widetilde{T}_q defined by (16) $\widetilde{T}_q = \{z \in \operatorname{Mat}(q \times q, \mathbb{C}) \mid z = x + iy, x \in H_q, y \in \overline{\Omega}_q, \det z \neq 0\}.$

Its interior is the classical *tube domain* over the cone Ω_q , namely

$$T_q = \{ z \in \operatorname{Mat}(q \times q, \mathbf{C}) \mid z = x + iy, \ y \in \Omega_q \}.$$

Let $G = GL(q, \mathbb{C})$ act on $Mat(q \times q, \mathbb{C})$ by (17) $(q, z) \longmapsto gzg^*$.

The spaces $H_q, \Omega_q, \overline{\Omega}_q$ are stable under this action, and hence \widetilde{T}_q and T_q are invariant subsets under this action. We investigate the orbits and describe a full set of invariants for this action.

There is a natural invariant associated to a $GL(q, \mathbb{C})$ -orbit. To any $z \in \widetilde{T}_q$, we associate its *angular matrix* defined by

(18)
$$a = a(z) = z^{*^{-1}}z$$
.

Then the matrix associated to gzg^* is $g^{*^{-1}}ag^*$, so that the angular matrix a(z) belongs to the same conjugacy class when z runs through a $GL(q, \mathbb{C})$ -orbit. As we shall see (Theorem 3.3 and Theorem 3.13), this invariant is close to characterizing the orbits.

Let us first prove some elementary properties of the angular matrix.

PROPOSITION 3.1. Let $z = x + iy \in \widetilde{T}_q$, and let $a = z^{*^{-1}}z$ be its angular matrix. Then

(i) $\operatorname{Sp}(a) \subset \operatorname{U}_1 = \{\mu \in \mathbf{C}, |\mu| = 1\};$

(ii) if $1 \in \text{Sp}(a)$, then y is degenerate and

$$\{v \in \mathbf{C}^q \mid av = v\} = \{v \in \mathbf{C}^q \mid yv = 0\}.$$

Proof. Let μ be an eigenvalue of a, and let $v \neq 0$ be an eigenvector for the eigenvalue μ . Then $zv = \mu z^* v$, and hence

$$(zv, v) = \mu(z^*v, v) = \mu(v, zv) = \mu(zv, v)$$

If $(zv, v) \neq 0$, then $|\mu| = 1$. So we now assume (zv, v) = 0. This amounts to (xv, v) + i(yv, v) = 0, so that in particular (yv, v) = 0. Now recall that y is positive semi-definite. So the condition (yv, v) = 0 implies that yv = 0. From this it follows that $zv = xv = z^*v$, and as z is assumed to be invertible, this implies $\mu = 1$. This shows (i) and part of (ii). Conversely, the condition yv = 0 implies trivially av = v. \Box In particular, we may consider the polynomial $d(\mu) = \det(z - \mu z^*)$. The roots of d are the eigenvalues of the angular matrix. The set of these roots, counted with their multiplicities, will be called the *angular spectrum* of z.

We first consider the case of T_q . So let $z = x + iy \in T_q$. Then as y is positive-definite, we may define its square root $y^{1/2}$ as the unique positive-definite Hermitian matrix whose square is y. Then we may write

$$x + iy = y^{\frac{1}{2}} (y^{-\frac{1}{2}} x y^{-\frac{1}{2}} + i 1_q) y^{\frac{1}{2}}.$$

This shows that any $GL(q, \mathbb{C})$ -orbit contains some element of the form $x+i1_q$, where $x \in H_q$. But by the classical diagonalization theorem for Hermitian forms, there exists an orthonormal basis in which the Hermitian form associated to x is diagonal. In other words, there exists a unitary matrix u and real numbers $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_q$ such that

$$uxu^* = \Lambda = egin{pmatrix} \lambda_1 & & & \ & \lambda_2 & & \ & & \ddots & \ & & & \ddots & \ & & & & \lambda_q \end{pmatrix}.$$

Moreover, if Λ and Λ' are two such diagonal matrices, then $\Lambda + i\mathbf{1}_q$ and $\Lambda' + i\mathbf{1}_q$ are not conjugate under the action of $GL(q, \mathbb{C})$ unless $\Lambda = \Lambda'$. Hence we have shown the following result, which of course is the well-known fact that there is a simultaneous diagonalization for two Hermitian forms if one of them is positive-definite.

THEOREM 3.2. The set of matrices of the form

(19)

$$\Lambda = \begin{pmatrix} \lambda_1 + i & & \\ & \lambda_2 + i & \\ & & \ddots & \\ & & & \lambda_q + i \end{pmatrix}$$

with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_q$ is a full set of representatives of the $GL(q, \mathbb{C})$ -orbits in T_q .

The angular matrix associated to Λ is

(20)
$$\begin{pmatrix} \frac{\lambda_1+i}{\lambda_1-i} & & \\ & \frac{\lambda_2+i}{\lambda_2-i} & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & \frac{\lambda_q+i}{\lambda_q-i} \end{pmatrix}.$$

The latter is a semi-simple matrix with spectral values

$$\mu_j = \frac{\lambda_j + i}{\lambda_j - i}$$

for $1 \le j \le q$. Observe that these spectral values are complex numbers of modulus 1, but always different from 1. From the u_j we may recover the λ_j by the formula

$$\lambda_j = i \, \frac{1+\mu_j}{1-\mu_j} \, .$$

From these observations we get the following result.

THEOREM 3.3. Two elements z and z' of T_q belong to the same $GL(q, \mathbb{C})$ -orbit if and only if their angular matrices are conjugate. The angular spectrum is a full set of invariants for the action of $GL(q, \mathbb{C})$ on T_q .

The situation for T_q is more complicated. In fact we may consider the extreme case where y = 0. Then x corresponds to a non-degenerate Hermitian form, and the orbit picture is given by the signature. So we need to consider matrices of the form

$$\Upsilon = \Upsilon_{n_+, n_-} = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & -1 & \\ & & & \ddots & \\ & & & & & -1 \end{pmatrix}$$

with n_+ diagonal entries equal to +1 and n_- diagonal entries equal to -1, n_+ and n_- being arbitrary nonnegative integers such that $n_+ + n_- = q$. The corresponding angular matrix is the identity matrix $\mathbf{1}_q$.

Another source of difficulty comes from the fact that it is not always possible to find a basis in which both Hermitian forms associated to x and y are diagonal. For instance if q = 2, consider the matrix

$$z = \begin{pmatrix} i & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix} + i \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Notice that its angular matrix is

$$a = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

which is not semisimple.

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From these examples we see that neither the angular spectrum of z nor the conjugacy class of the angular matrix characterizes the orbit of z.

Let n_1, n_2, n_3, n_4 be four nonnegative integers such that $n_1+2n_2+n_3+n_4 = q$, and let $\lambda_1, \lambda_2, \ldots, \lambda_{n_1}$ be n_1 real numbers satisfying the condition

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1}$$
.

To such data we associate the matrix $\Lambda = \Lambda(\lambda_1, \lambda_2, \dots, \lambda_{n_1}, n_2, n_3, n_4)$ given by

i 1

1 0

1

-1 ·.

1

(21)

where there are n_2 diagonal 2 × 2 submatrices of the form $\begin{pmatrix} l & 1 \\ 1 & 0 \end{pmatrix}$, n_3 diagonal terms equal to 1 and n_4 diagonal terms equal to -1.

THEOREM 3.4. Any $GL(q, \mathbb{C})$ orbit in \widetilde{T}_q contains one and only one matrix of the form $\Lambda(\lambda_1, \lambda_2, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$.

Before beginning the proof, let us prove a couple of lemmas. Lemmas 3.6 and 3.7 are related to the classical Gauss's algorithm for diagonalizing an Hermitian form. Let r, s, n be three nonnegative integers such that r + s = n.

LEMMA 3.5. The stabilizer in GL(n, C) of the matrix $y_r = \begin{pmatrix} \mathbf{1}_r \\ \mathbf{0}_s \end{pmatrix}$ is the subgroup

(22)
$$G_r = \left\{ \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \right\}$$

where $u \in U(r)$, $v \in Mat(r, s)$, $h \in GL(s, \mathbb{C})$.

 $\lambda_{n_1}+i \ egin{array}{ccc} i & 1 \ 1 & 0 \end{array}$

Proof. Easy computation.

Now we study the action of G_r in H_n . If $x \in H_n$, let us write

$$x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix}$$

where $\alpha \in H_r, b \in \operatorname{Mat}(r \times s, \mathbb{C})$ and $\gamma \in H_s$. If $g = \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \in G_r$, then $gxg^* = \begin{pmatrix} \alpha' & b' \\ b'^* & \gamma' \end{pmatrix}$, with

$$\alpha' = u\alpha u^* + ubv^* + vb^*u^* + v\gamma v^*$$
$$b' = ubh^* + v\gamma h^*$$
$$\gamma' = h\gamma h^* .$$

LEMMA 3.6. Let $x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix} \in H_n$, with $\alpha \in H_r$, $b \in Mat(r \times s, \mathbb{C})$ and $\gamma \in H_s$. Assume det $\gamma \neq 0$. Then the orbit of x under G_r contains a matrix of the form $\begin{pmatrix} \alpha' & 0 \\ 0 & \gamma \end{pmatrix}$ with $\alpha' \in H_r$.

Proof. This is a consequence of the previous formula with $u = \mathbf{1}_r$, $v = -b\gamma^{-1}$ and $h = \mathbf{1}_s$.

LEMMA 3.7. Let $x = \begin{pmatrix} \alpha & b \\ b^* & 0 \end{pmatrix} \in H_n$, with rank b = s (so in particular $r \geq s$). Then the orbit of x under G_r contains an element of the form

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

with $\beta \in H_{r-s}$.

Proof. Consider the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & h \end{pmatrix}, u \in U(r), h \in GL_s(\mathbb{C}) \right\}$. It acts on the component b by $b' = ubh^*$. As $\operatorname{rank}(b) = s$, we may think of b as a set of s independent vectors in \mathbb{C}^r . By the Gram-Schmidt process, it is possible to find $h \in GL_s(\mathbb{C})$ such that bh^* is a s-orthonormal frame in \mathbb{C}^r . But now two such frames are conjugate by the (left) action of U(r). Hence there exists $u \in U(r)$ such that

$$ubh^* = \begin{pmatrix} 0 \\ \mathbf{1}_s \end{pmatrix}.$$

The matrix x we started with is conjugate under G_r to a matrix of the form

$$egin{pmatrix} lpha' & c & 0 \ c^* & eta & \mathbf{l}_s \ 0 & \mathbf{l}_s & 0 \end{pmatrix}$$

where $\alpha' \in H_{r-s}$, $\beta \in H_s$ and $c \in Mat((r-s) \times s, \mathbb{C})$. Now we use the action of the element

$$g = \begin{pmatrix} \mathbf{1}_{r-s} & 0 & -c \\ 0 & \mathbf{1}_s & -\frac{\beta}{2} \\ 0 & 0 & \mathbf{1}_s \end{pmatrix} \in G_r$$

to get the result. \Box

We are now ready to start the proof of Theorem 3.4.

STEP 1. Let $z = x + iy \in \widetilde{T}_q$. As y is positive semidefinite, there exists an element $g \in GL(q, \mathbb{C})$ such that

$$gyg^* = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with r diagonal entries equal to 1, and s diagonal entries equal to 0, r and s being nonnegative integers satisfying r + s = q. In other terms, any $GL(q, \mathbf{C})$ -orbit in \tilde{T}_q contains an element of the form

$$\begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}$$

with $\alpha \in H_r, \gamma \in H_s, b \in Mat(r \times s, \mathbb{C})$.

STEP 2. Now assume x is of the form

$$x = \begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}.$$

Consider γ . It is an Hermitian matrix of size *s*, and under the action of $GL(s, \mathbb{C})$ it can be transformed to

$$\begin{pmatrix} \mathbf{0}_{n_2} & 0 & 0 \\ 0 & \mathbf{1}_{n_3} & 0 \\ 0 & 0 & -\mathbf{1}_{n_4} \end{pmatrix}$$

where $n_2 + n_3 + n_4 = s$. Hence x is conjugate under the action of G_r to an element of the form

$$egin{pmatrix} lpha & b' & c' \ b'^* & 0 & 0 \ c'^* & 0 & \Upsilon \end{pmatrix}$$

where $\alpha \in H_r$, $b' \in Mat(r \times n_2, \mathbb{C})$, $c' \in Mat(r \times (n_3 + n_4), \mathbb{C})$ and

$$\Upsilon = egin{pmatrix} \mathbf{1}_{n_3} & 0 \ 0 & -\mathbf{1}_{n_4} \end{pmatrix}.$$

Using Lemma 3.6, we see that x is conjugate under the action of G_s to an element of the form

$$egin{pmatrix} lpha'' & b'' & 0 \ b''^* & 0 & 0 \ 0 & 0 & \Upsilon \end{pmatrix},$$

with $\alpha'' \in H_r$, $b'' \in Mat(r \times n_2, \mathbb{C})$.

STEP 3. Assume now that

$$x = \begin{pmatrix} \alpha & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

with $\alpha \in H_r$ and $b \in Mat(r \times n_2, \mathbb{C})$. Recall that

$$x + iy = \begin{pmatrix} \alpha + i\mathbf{1}_r & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

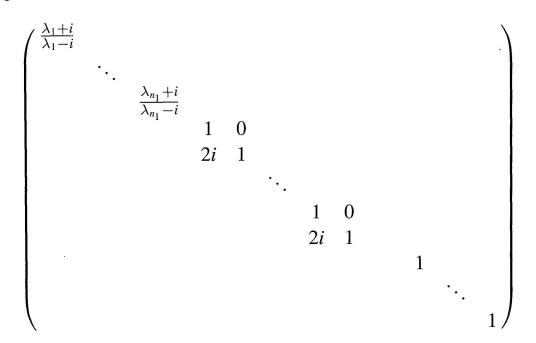
is assumed to be invertible. This shows that $rank(b) = n_2$. So we may apply Lemma 3.7 to see that x is conjugate under G_r to an element of the form

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \Upsilon \end{pmatrix}$$

with $\beta \in H_{r-n_2}$.

STEP 4. Set $n_1 = r - n_2$. The last step is just to put the element $\beta \in H_{n_1}$ in diagonal form under the action of $U(n_1)$. Up to minor rearrangements of the matrix, this shows that any $GL(q, \mathbb{C})$ -orbit in \widetilde{T}_q contains an element of the form $\Lambda(\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$.

STEP 5. It remains to show that two Λ 's are not conjugate under $GL(q, \mathbb{C})$. The angular matrix associated to $\Lambda(\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$ is



where there are $n_2 \ 2 \times 2$ submatrices $\begin{pmatrix} 1 & 0 \\ 2i & 1 \end{pmatrix}$, and $n_3 + n_4$ diagonal elements equal to 1. From the Jordan normal form theorem, we deduce that if $\Lambda(\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$ and $\Lambda(\lambda'_1, \ldots, \lambda'_{n_1}, n'_2, n'_3, n'_4)$ are in a same $GL(q, \mathbb{C})$ -orbit, then $n_1 = n'_1$, $\lambda_j = \lambda'_j$ for all $j, 1 \le j \le n_1$, $n_2 = n'_2$ and $n_3 + n_4 = n'_3 + n'_4$. Now the matrix $\Lambda(\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4) = L + iM$ and $\Lambda' = L' + iM'$, with $L, L', M, M' \in H_n$. As Λ and Λ' are supposed to be in the same $GL(q, \mathbb{C})$ -orbit, L and L' are also in the same $GL(q, \mathbb{C})$ -orbit, and so they must have the same signature. This forces $n_3 = n'_3$ and $n_4 = n'_4$, and hence $\Lambda = \Lambda'$.

We can now give the solution to the orbit problem we addressed at the end of Section 2. Recall that for any integer r such that $0 \le r \le q$ we defined

$$\widetilde{T}_q^{(r)} = \{ z = x + iy \mid y \in \overline{\Omega}_q, \text{ rank}(y) \le r, z \text{ invertible} \}.$$

LEMMA 3.8. Let n_1, n_2, n_3, n_4 be four integers such that

$$n_1 + 2n_2 + n_3 + n_4 = q \,,$$

and let $\lambda_1, \ldots, \lambda_{n_1}$ be n_1 real numbers. Then the standard matrix $\Lambda = \Lambda(\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$ belongs to $\widetilde{T}_q^{(r)}$ if and only if $n_1 + n_2 \leq r$.

In fact the rank of $\frac{1}{2i}(\Lambda - \Lambda^*)$ is $n_1 + n_2$.

THEOREM 3.9. Any $GL(q, \mathbb{C})$ -orbit in $\widetilde{T}_q^{(r)}$ contains a unique standard matrix $\Lambda((\lambda_1, \ldots, \lambda_{n_1}, n_2, n_3, n_4)$ with $n_1 + n_2 \leq r$.

We now want an analog of Theorem 3.3. As we have already noticed, the conjugacy class of the angular matrix does not determine the orbit of the matrix. We need a finer invariant, which we will construct now.

LEMMA 3.10. The space \tilde{T}_q is connected and simply connected.

Proof. As T_q is connected and $T_q \subset \widetilde{T}_q \subset \overline{T}_q$, the space \widetilde{T}_q is connected. Take $i\mathbf{1}_q$ as base point in \widetilde{T}_q , and observe that for any $z \in \widetilde{T}_q$ and any s > 0, $z + is\mathbf{1}_q$ is in T_q . So if $(\gamma(t), t \in [0, 1])$ is a path in \widetilde{T}_q starting and ending at $i\mathbf{1}_q$ then we can deform it by homotopy to $\gamma_s(t) = \gamma(t) + is(s - 1)\mathbf{1}_q$, which for s > 0 is a path inside T_q . But T_q as a tube-type domain is simply connected. \Box

The function $z \mapsto \det(z)$ is a continuous function from \widetilde{T}_q into \mathbb{C}^* . From Lemma 3.10, there exists a unique continuous determination of the argument of $\det(z)$ denoted by $\arg \det: \widetilde{T}_q \longrightarrow \mathbb{R}$ such that $\arg \det i\mathbf{1}_q = q\frac{\pi}{2}$. If $Y \in \Omega_q$, then $\arg \det iy = q\frac{\pi}{2}$. If $z \in \widetilde{T}_q$ and $g \in \operatorname{GL}(q, \mathbb{C})$, then $\det gzg^* = |\det g|^2 \det z$, and $gi\mathbf{1}_qg^* = igg^* \in i\Omega_q$, so that

 $\arg \det gzg^* = \arg \det z$.

This provides a new invariant for the action of $GL(q, \mathbb{C})$ on \widetilde{T}_q .

LEMMA 3.11. Let $\Lambda = \Lambda(\lambda_1, ..., \lambda_{n_1}, n_2, n_3, n_4)$. Then

(23) $\arg \det \Lambda = \arg(\lambda_1 + i) + \dots + \arg(\lambda_{n_1} + i) + n_2 \pi + n_4 \pi$

where arg is used for the principal determination of the argument of a non-zero complex number.

Proof. We need to describe a continuous path from $i\mathbf{1}_q$ to Λ inside \widetilde{T}_q . For clarity of exposition, we describe successively the path for each diagonal block (either a one-dimensional or a two-dimensional submatrix) of Λ , and compute the contribution of each block to the function arg det.

For a block of the form $\lambda + i$, with $\lambda \in \mathbf{R}$ we use the path $t \mapsto t\lambda + i$, $0 \le t \le 1$, and so the contribution of this block is $\arg(\lambda + i)$.

For a block of the form $\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$, we use the path

$$t \mapsto \begin{pmatrix} i & t \\ t & i(1-t^2) \end{pmatrix}, \ 0 \le t \le 1.$$

The corresponding determinant of this 2×2 -block is constant along the path and equal to -1. Hence the contribution of this block is $2\frac{\pi}{2} = \pi$.

For a block of the form 1, we use the path $t \mapsto e^{i\frac{\pi}{2}(1-t)}$, $0 \le t \le 1$, and we see that the corresponding contribution is 0.

For a block of the form -1, we use the path $t \mapsto e^{i\frac{\pi}{2}(1+t)}$, $0 \le t \le 1$, and we see that the corresponding contribution is π .

Putting together the contribution of the blocks, we get the result. \Box

COROLLARY 3.12. Let Λ and Λ' be two standard matrices. Assume that their angular matrices coincide and that $\operatorname{arg} \operatorname{det} \Lambda = \operatorname{arg} \operatorname{det} \Lambda'$. Then $\Lambda = \Lambda'$.

Proof. In fact we noticed that the equality of angular matrices implies the equality of the parameters except for $n_3 = n'_3$ and $n_4 = n'_4$. But from (23), we see that the equality of the determination of the arguments of the determinants implies $n_4 = n'_4$ (and hence $n_3 = n'_3$). \Box

Now we can state the conclusion of this section, which is a consequence of Theorem 3.4 and Corollary 3.12.

THEOREM 3.13. Let $z, z' \in \tilde{T}_q$, and assume that the angular matrices of z and z' are conjugate, and that $\arg \det z = \arg \det z'$. Then z and z' belong to the same orbit under the action of $GL(q, \mathbb{C})$.

REMARK. Let $z \in \widetilde{T}_q$. Let $a = z^{*^{-1}}z$. Then

$$\det a = \frac{\det z}{\det z} = |\det z|^{-2} (\det z)^2.$$

So 2 arg det z is a determination of $\arg(\det a)$. If z and z' are two matrices in \widetilde{T}_q with the same angular matrix, then $\arg \det z$ and $\arg \det z'$ differ by an integral multiple of π . So the new invariant needed to characterize the orbits under $\operatorname{GL}(q, \mathbb{C})$ has to be regarded as a Z-valued function. In this sense, it is a generalization of the signature.