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Autor: Clerc, Jean-Louis
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3. ORBITS FOR THE GL_q -ACTION ON \tilde{T}_q

Any $z \in \text{Mat}(q \times q, \mathbf{C})$ can be written in a unique way as $z = x + iy$, with $x, y \in H_q$. We will be concerned with the set \tilde{T}_q defined by

$$(16) \quad \tilde{T}_q = \{z \in \text{Mat}(q \times q, \mathbf{C}) \mid z = x + iy, x \in H_q, y \in \overline{\Omega}_q, \det z \neq 0\}.$$

Its interior is the classical *tube domain* over the cone Ω_q , namely

$$T_q = \{z \in \text{Mat}(q \times q, \mathbf{C}) \mid z = x + iy, y \in \Omega_q\}.$$

Let $G = GL(q, \mathbf{C})$ act on $\text{Mat}(q \times q, \mathbf{C})$ by

$$(17) \quad (g, z) \longmapsto gzg^*.$$

The spaces $H_q, \Omega_q, \overline{\Omega}_q$ are stable under this action, and hence \tilde{T}_q and T_q are invariant subsets under this action. We investigate the orbits and describe a full set of invariants for this action.

There is a natural invariant associated to a $GL(q, \mathbf{C})$ -orbit. To any $z \in \tilde{T}_q$, we associate its *angular matrix* defined by

$$(18) \quad a = a(z) = z^{*-1}z.$$

Then the matrix associated to gzg^* is $g^{*-1}ag^*$, so that the angular matrix $a(z)$ belongs to the same conjugacy class when z runs through a $GL(q, \mathbf{C})$ -orbit. As we shall see (Theorem 3.3 and Theorem 3.13), this invariant is close to characterizing the orbits.

Let us first prove some elementary properties of the angular matrix.

PROPOSITION 3.1. *Let $z = x + iy \in \tilde{T}_q$, and let $a = z^{*-1}z$ be its angular matrix. Then*

- (i) $\text{Sp}(a) \subset U_1 = \{\mu \in \mathbf{C}, |\mu| = 1\}$;
- (ii) if $1 \in \text{Sp}(a)$, then y is degenerate and

$$\{v \in \mathbf{C}^q \mid av = v\} = \{v \in \mathbf{C}^q \mid yv = 0\}.$$

Proof. Let μ be an eigenvalue of a , and let $v \neq 0$ be an eigenvector for the eigenvalue μ . Then $zv = \mu z^*v$, and hence

$$(zv, v) = \mu(z^*v, v) = \mu(v, zv) = \mu(\overline{zv, v}).$$

If $(zv, v) \neq 0$, then $|\mu| = 1$. So we now assume $(zv, v) = 0$. This amounts to $(xv, v) + i(yv, v) = 0$, so that in particular $(yv, v) = 0$. Now recall that y is positive semi-definite. So the condition $(yv, v) = 0$ implies that $yv = 0$. From this it follows that $zv = xv = z^*v$, and as z is assumed to be invertible, this implies $\mu = 1$. This shows (i) and part of (ii). Conversely, the condition $yv = 0$ implies trivially $av = v$. \square

In particular, we may consider the polynomial $d(\mu) = \det(z - \mu z^*)$. The roots of d are the eigenvalues of the angular matrix. The set of these roots, counted with their multiplicities, will be called the *angular spectrum* of z .

We first consider the case of T_q . So let $z = x + iy \in T_q$. Then as y is positive-definite, we may define its square root $y^{1/2}$ as the unique positive-definite Hermitian matrix whose square is y . Then we may write

$$x + iy = y^{\frac{1}{2}}(y^{-\frac{1}{2}}xy^{-\frac{1}{2}} + i\mathbf{1}_q)y^{\frac{1}{2}}.$$

This shows that any $GL(q, \mathbf{C})$ -orbit contains some element of the form $x + i\mathbf{1}_q$, where $x \in H_q$. But by the classical diagonalization theorem for Hermitian forms, there exists an orthonormal basis in which the Hermitian form associated to x is diagonal. In other words, there exists a unitary matrix u and real numbers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q$ such that

$$uxu^* = \Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_q \end{pmatrix}.$$

Moreover, if Λ and Λ' are two such diagonal matrices, then $\Lambda + i\mathbf{1}_q$ and $\Lambda' + i\mathbf{1}_q$ are not conjugate under the action of $GL(q, \mathbf{C})$ unless $\Lambda = \Lambda'$. Hence we have shown the following result, which of course is the well-known fact that there is a simultaneous diagonalization for two Hermitian forms if one of them is positive-definite.

THEOREM 3.2. *The set of matrices of the form*

$$(19) \quad \Lambda = \begin{pmatrix} \lambda_1 + i & & & \\ & \lambda_2 + i & & \\ & & \ddots & \\ & & & \lambda_q + i \end{pmatrix}$$

with $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_q$ is a full set of representatives of the $GL(q, \mathbf{C})$ -orbits in T_q .

The angular matrix associated to Λ is

$$(20) \quad \begin{pmatrix} \frac{\lambda_1+i}{\lambda_1-i} & & & \\ & \frac{\lambda_2+i}{\lambda_2-i} & & \\ & & \ddots & \\ & & & \frac{\lambda_q+i}{\lambda_q-i} \end{pmatrix}.$$

Now we study the action of G_r in H_n . If $x \in H_n$, let us write

$$x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix}$$

where $\alpha \in H_r$, $b \in \text{Mat}(r \times s, \mathbf{C})$ and $\gamma \in H_s$. If $g = \begin{pmatrix} u & v \\ 0 & h \end{pmatrix} \in G_r$, then $gxg^* = \begin{pmatrix} \alpha' & b' \\ b'^* & \gamma' \end{pmatrix}$, with

$$\alpha' = u\alpha u^* + ubv^* + vb^*u^* + v\gamma v^*$$

$$b' = ubh^* + v\gamma h^*$$

$$\gamma' = h\gamma h^*.$$

LEMMA 3.6. Let $x = \begin{pmatrix} \alpha & b \\ b^* & \gamma \end{pmatrix} \in H_n$, with $\alpha \in H_r$, $b \in \text{Mat}(r \times s, \mathbf{C})$ and $\gamma \in H_s$. Assume $\det \gamma \neq 0$. Then the orbit of x under G_r contains a matrix of the form $\begin{pmatrix} \alpha' & 0 \\ 0 & \gamma \end{pmatrix}$ with $\alpha' \in H_r$.

Proof. This is a consequence of the previous formula with $u = \mathbf{1}_r$, $v = -b\gamma^{-1}$ and $h = \mathbf{1}_s$.

LEMMA 3.7. Let $x = \begin{pmatrix} \alpha & b \\ b^* & 0 \end{pmatrix} \in H_n$, with $\text{rank } b = s$ (so in particular $r \geq s$). Then the orbit of x under G_r contains an element of the form

$$\begin{pmatrix} \beta & 0 & 0 \\ 0 & 0 & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

with $\beta \in H_{r-s}$.

Proof. Consider the subgroup $\left\{ \begin{pmatrix} u & 0 \\ 0 & h \end{pmatrix}, u \in U(r), h \in GL_s(\mathbf{C}) \right\}$. It acts on the component b by $b' = ubh^*$. As $\text{rank}(b) = s$, we may think of b as a set of s independent vectors in \mathbf{C}^r . By the Gram-Schmidt process, it is possible to find $h \in GL_s(\mathbf{C})$ such that bh^* is a s -orthonormal frame in \mathbf{C}^r . But now two such frames are conjugate by the (left) action of $U(r)$. Hence there exists $u \in U(r)$ such that

$$ubh^* = \begin{pmatrix} 0 \\ \mathbf{1}_s \end{pmatrix}.$$

The matrix x we started with is conjugate under G_r to a matrix of the form

$$\begin{pmatrix} \alpha' & c & 0 \\ c^* & \beta & \mathbf{1}_s \\ 0 & \mathbf{1}_s & 0 \end{pmatrix}$$

where $\alpha' \in H_{r-s}$, $\beta \in H_s$ and $c \in \text{Mat}((r-s) \times s, \mathbf{C})$. Now we use the action of the element

$$g = \begin{pmatrix} \mathbf{1}_{r-s} & 0 & -c \\ 0 & \mathbf{1}_s & -\frac{\beta}{2} \\ 0 & 0 & \mathbf{1}_s \end{pmatrix} \in G_r$$

to get the result. \square

We are now ready to start the proof of Theorem 3.4.

STEP 1. Let $z = x + iy \in \tilde{T}_q$. As y is positive semidefinite, there exists an element $g \in \text{GL}(q, \mathbf{C})$ such that

$$gyg^* = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix},$$

with r diagonal entries equal to 1, and s diagonal entries equal to 0, r and s being nonnegative integers satisfying $r + s = q$. In other terms, any $\text{GL}(q, \mathbf{C})$ -orbit in \tilde{T}_q contains an element of the form

$$\begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}$$

with $\alpha \in H_r, \gamma \in H_s, b \in \text{Mat}(r \times s, \mathbf{C})$.

STEP 2. Now assume x is of the form

$$x = \begin{pmatrix} \alpha + i\mathbf{1}_r & b \\ b^* & \gamma \end{pmatrix}.$$

Consider γ . It is an Hermitian matrix of size s , and under the action of $\text{GL}(s, \mathbf{C})$ it can be transformed to

$$\begin{pmatrix} \mathbf{0}_{n_2} & 0 & 0 \\ 0 & \mathbf{1}_{n_3} & 0 \\ 0 & 0 & -\mathbf{1}_{n_4} \end{pmatrix}$$

where $n_2 + n_3 + n_4 = s$. Hence x is conjugate under the action of G_r to an element of the form

$$\begin{pmatrix} \alpha & b' & c' \\ b'^* & 0 & 0 \\ c'^* & 0 & \Upsilon \end{pmatrix}$$

where $\alpha \in H_r$, $b' \in \text{Mat}(r \times n_2, \mathbf{C})$, $c' \in \text{Mat}(r \times (n_3 + n_4), \mathbf{C})$ and

$$\Upsilon = \begin{pmatrix} \mathbf{1}_{n_3} & 0 \\ 0 & -\mathbf{1}_{n_4} \end{pmatrix}.$$

Using Lemma 3.6, we see that x is conjugate under the action of G_s to an element of the form

$$\begin{pmatrix} \alpha'' & b'' & 0 \\ b''^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix},$$

with $\alpha'' \in H_r$, $b'' \in \text{Mat}(r \times n_2, \mathbf{C})$.

STEP 3. Assume now that

$$x = \begin{pmatrix} \alpha & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

with $\alpha \in H_r$ and $b \in \text{Mat}(r \times n_2, \mathbf{C})$. Recall that

$$x + iy = \begin{pmatrix} \alpha + i\mathbf{1}_r & b & 0 \\ b^* & 0 & 0 \\ 0 & 0 & \Upsilon \end{pmatrix}$$

is assumed to be invertible. This shows that $\text{rank}(b) = n_2$. So we may apply Lemma 3.7 to see that x is conjugate under G_r to an element of the form

$$\begin{pmatrix} \beta & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{n_2} & 0 \\ 0 & \mathbf{1}_{n_2} & 0 & 0 \\ 0 & 0 & 0 & \Upsilon \end{pmatrix}$$

with $\beta \in H_{r-n_2}$.

STEP 4. Set $n_1 = r - n_2$. The last step is just to put the element $\beta \in H_{n_1}$ in diagonal form under the action of $U(n_1)$. Up to minor rearrangements of the matrix, this shows that any $\text{GL}(q, \mathbf{C})$ -orbit in \tilde{T}_q contains an element of the form $\Lambda(\lambda_1, \dots, \lambda_{n_1}, n_2, n_3, n_4)$.

THEOREM 3.9. *Any $\text{GL}(q, \mathbf{C})$ -orbit in $\widetilde{T}_q^{(r)}$ contains a unique standard matrix $\Lambda((\lambda_1, \dots, \lambda_{n_1}, n_2, n_3, n_4))$ with $n_1 + n_2 \leq r$.*

We now want an analog of Theorem 3.3. As we have already noticed, the conjugacy class of the angular matrix does not determine the orbit of the matrix. We need a finer invariant, which we will construct now.

LEMMA 3.10. *The space \widetilde{T}_q is connected and simply connected.*

Proof. As T_q is connected and $T_q \subset \widetilde{T}_q \subset \overline{T}_q$, the space \widetilde{T}_q is connected. Take $i\mathbf{1}_q$ as base point in \widetilde{T}_q , and observe that for any $z \in \widetilde{T}_q$ and any $s > 0$, $z + is\mathbf{1}_q$ is in T_q . So if $(\gamma(t), t \in [0, 1])$ is a path in \widetilde{T}_q starting and ending at $i\mathbf{1}_q$ then we can deform it by homotopy to $\gamma_s(t) = \gamma(t) + is(s-1)\mathbf{1}_q$, which for $s > 0$ is a path inside T_q . But T_q as a tube-type domain is simply connected. \square

The function $z \mapsto \det(z)$ is a continuous function from \widetilde{T}_q into \mathbf{C}^* . From Lemma 3.10, there exists a unique continuous determination of the argument of $\det(z)$ denoted by $\arg \det: \widetilde{T}_q \rightarrow \mathbf{R}$ such that $\arg \det i\mathbf{1}_q = q\frac{\pi}{2}$. If $Y \in \Omega_q$, then $\arg \det iy = q\frac{\pi}{2}$. If $z \in \widetilde{T}_q$ and $g \in \text{GL}(q, \mathbf{C})$, then $\det gzg^* = |\det g|^2 \det z$, and $gi\mathbf{1}_qg^* = igg^* \in i\Omega_q$, so that

$$\arg \det gzg^* = \arg \det z.$$

This provides a new invariant for the action of $\text{GL}(q, \mathbf{C})$ on \widetilde{T}_q .

LEMMA 3.11. *Let $\Lambda = \Lambda(\lambda_1, \dots, \lambda_{n_1}, n_2, n_3, n_4)$. Then*

$$(23) \quad \arg \det \Lambda = \arg(\lambda_1 + i) + \dots + \arg(\lambda_{n_1} + i) + n_2\pi + n_4\pi$$

where \arg is used for the principal determination of the argument of a non-zero complex number.

Proof. We need to describe a continuous path from $i\mathbf{1}_q$ to Λ inside \widetilde{T}_q . For clarity of exposition, we describe successively the path for each diagonal block (either a one-dimensional or a two-dimensional submatrix) of Λ , and compute the contribution of each block to the function $\arg \det$.

For a block of the form $\lambda + i$, with $\lambda \in \mathbf{R}$ we use the path $t \mapsto t\lambda + i$, $0 \leq t \leq 1$, and so the contribution of this block is $\arg(\lambda + i)$.

For a block of the form $\begin{pmatrix} i & 1 \\ 1 & 0 \end{pmatrix}$, we use the path

$$t \mapsto \begin{pmatrix} i & t \\ t & i(1-t^2) \end{pmatrix}, \quad 0 \leq t \leq 1.$$

The corresponding determinant of this 2×2 -block is constant along the path and equal to -1 . Hence the contribution of this block is $2\frac{\pi}{2} = \pi$.

For a block of the form 1 , we use the path $t \mapsto e^{i\frac{\pi}{2}(1-t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is 0 .

For a block of the form -1 , we use the path $t \mapsto e^{i\frac{\pi}{2}(1+t)}$, $0 \leq t \leq 1$, and we see that the corresponding contribution is π .

Putting together the contribution of the blocks, we get the result. \square

COROLLARY 3.12. *Let Λ and Λ' be two standard matrices. Assume that their angular matrices coincide and that $\arg \det \Lambda = \arg \det \Lambda'$. Then $\Lambda = \Lambda'$.*

Proof. In fact we noticed that the equality of angular matrices implies the equality of the parameters except for $n_3 = n'_3$ and $n_4 = n'_4$. But from (23), we see that the equality of the determination of the arguments of the determinants implies $n_4 = n'_4$ (and hence $n_3 = n'_3$). \square

Now we can state the conclusion of this section, which is a consequence of Theorem 3.4 and Corollary 3.12.

THEOREM 3.13. *Let $z, z' \in \tilde{T}_q$, and assume that the angular matrices of z and z' are conjugate, and that $\arg \det z = \arg \det z'$. Then z and z' belong to the same orbit under the action of $\mathrm{GL}(q, \mathbf{C})$.*

REMARK. Let $z \in \tilde{T}_q$. Let $a = z^{*-1}z$. Then

$$\det a = \frac{\det z}{\det z} = |\det z|^{-2}(\det z)^2.$$

So $2 \arg \det z$ is a determination of $\arg(\det a)$. If z and z' are two matrices in \tilde{T}_q with the same angular matrix, then $\arg \det z$ and $\arg \det z'$ differ by an integral multiple of π . So the new invariant needed to characterize the orbits under $\mathrm{GL}(q, \mathbf{C})$ has to be regarded as a \mathbf{Z} -valued function. In this sense, it is a generalization of the signature.