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THE HILBERT METRIC AND GROMOV HYPERBOLICITY

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ABSTRACT. We give some sufficient conditions for Hilbert's metric on convex domains D to be Gromov hyperbolic. The conditions involve an intersecting chords property, which we in turn relate to the Menger curvature of triples of boundary points and, in the case the boundary is smooth, to differential geometric curvature of ∂D. In particular, the intersecting chords property and hence Gromov hyperbolicity is established for bounded, convex C^2-domains in $\mathbf{R}^n$ with non-zero curvature.

We also give some necessary conditions for hyperbolicity: the boundary must be of class C^1 and may not contain a line segment. Furthermore we prove a statement about the asymptotic geometry of the Hilbert metric on arbitrary convex (i.e. not necessarily strictly convex) bounded domains, with an application to maps which do not increase Hilbert distance.

INTRODUCTION

Let D be a bounded convex domain in $\mathbf{R}^n$ and let h be the Hilbert metric, which is defined as follows. For any distinct points $x, y \in D$, let x' and y' be the intersections of the line through x and y with ∂D closest to x and y respectively. Then

$$h(x,y) = \log \frac{yx' \cdot xy'}{xx' \cdot yy'}$$

where zw denotes the Euclidean distance $\|z - w\|$ between two points. The expression $\frac{yx' \cdot xy'}{xx' \cdot yy'}$ is called the *cross-ratio* of four collinear points and is invariant under projective transformations. For the basic properties of the distance h we refer to [Bu55] or [dlH93].

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