# Introduction

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# THE HILBERT METRIC AND GROMOV HYPERBOLICITY

by Anders KARLSSON[1]) and Guennadi A. NOSKOV[2])

ABSTRACT. We give some sufficient conditions for Hilbert's metric on convex domains $D$ to be Gromov hyperbolic. The conditions involve an intersecting chords property, which we in turn relate to the Menger curvature of triples of boundary points and, in the case the boundary is smooth, to differential geometric curvature of $\partial D$. In particular, the intersecting chords property and hence Gromov hyperbolicity is established for bounded, convex $C^2$-domains in $\mathbf{R}^n$ with non-zero curvature.

We also give some necessary conditions for hyperbolicity: the boundary must be of class $C^1$ and may not contain a line segment. Furthermore we prove a statement about the asymptotic geometry of the Hilbert metric on arbitrary convex (i.e. not necessarily strictly convex) bounded domains, with an application to maps which do not increase Hilbert distance.

## INTRODUCTION

Let $D$ be a bounded convex domain in $\mathbf{R}^n$ and let $h$ be the Hilbert metric, which is defined as follows. For any distinct points $x, y \in D$, let $x'$ and $y'$ be the intersections of the line through $x$ and $y$ with $\partial D$ closest to $x$ and $y$ respectively. Then

$$h(x,y) = \log \frac{yx' \cdot xy'}{xx' \cdot yy'}$$

where $zw$ denotes the Euclidean distance $\|z - w\|$ between two points. The expression $\frac{yx' \cdot xy'}{xx' \cdot yy'}$ is called the *cross-ratio* of four collinear points and is invariant under projective transformations. For the basic properties of the distance $h$ we refer to [Bu55] or [dlH93].

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We will give here some sufficient conditions for the metric space $(D, h)$ to be hyperbolic in the sense of Gromov. Namely, we show that a certain intersecting chords property implies Gromov hyperbolicity (Theorem 2.1). This intersecting chords property holds when the (Menger) curvature of any three points of the domain's boundary is uniformly bounded from both above and below in a certain way (Corollary 1.2). Domains with $C^2$ boundary of everywhere nonzero curvature satisfy this condition as will be proved in Section 3. Beardon showed in [Be97] (see also [Be99]) that a weaker intersecting chords property holds for any bounded strictly convex domain and he used this to establish some weak hyperbolicity results for the Hilbert metric. In Section 5 we prove a generalization of his results to any bounded convex domain.

It would also be interesting to understand what conditions on $\partial D$ are necessary in order for $(D, h)$ to be Gromov hyperbolic. For example, in Section 4 we give an argument showing that $\partial D$ must be of class $C^1$ and that it may not contain a line segment.

Some parts of the results in this paper might already be known: Y. Benoist has told us that a convex domain with $C^2$ boundary is Gromov hyperbolic if the curvature of the boundary is everywhere nonzero. Benoist has also found examples of Gromov hyperbolic Hilbert geometries whose boundaries are $C^1$ but not $C^2$. In [Be99] it is mentioned that C. Bell has proved an intersecting chords theorem in an unpublished work. However, we have not found the present arguments or our main results in the literature. Furthermore, we have not found the simple and attractive Proposition 1.1 and Corollary 3.5 stated or discussed anywhere, although these facts are most likely known. They should belong to ancient Greek geometry and classical differential geometry respectively.

Since the Hilbert distance can be defined in analogy with Kobayashi's pseudo-distance on complex spaces [Ko84], we would like to mention that Balogh and Bonk proved in [BB00] that the Kobayashi metric on any bounded strictly pseudoconvex domain with $C^2$ boundary is Gromov hyperbolic.

Note that metric spaces of this type are CAT(0) only in exceptional cases (see [BH99] for the definition). Indeed, Kelly and Straus proved in [KS58] that if $(D, h)$ is nonpositively curved in the sense of Busemann then $D$ is an ellipsoid and hence $(D, h)$ is the $n$-dimensional hyperbolic space. Compare this to the situation for Banach spaces: a Banach space is CAT(0) if and only if it is a Hilbert space. Another category of results is of the following type: if $D$ has a large (infinite, cocompact, etc.) automorphism group and $C^2$-smooth boundary, then it is an ellipsoid; see the work of Socié-Méthou [SM00].

The Hilbert metric has found several applications, see [Bi57], [Li95] and [Me95] just to mention a few instances. Typically the idea is to apply the contraction mapping principle to maps which do not increase Hilbert distances (e.g. affine maps).

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## 1. Intersecting chords in convex domains

From elementary school we know that if $c_1, c_2$ are two intersecting chords in a circle, then $l_1 l_1' = l_2 l_2'$ where $l_1, l_1'$ and $l_2, l_2'$ denote the respective lengths of the segments into which the two chords are divided. (This follows immediately from the similarity of the associated triangles, see Fig. 1.)

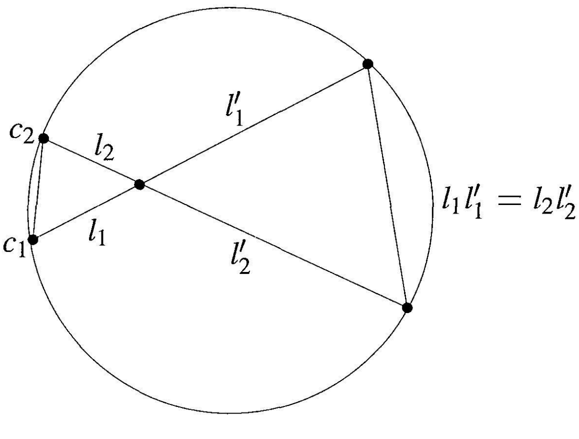


FIGURE 1
Intersecting chords in a circle

A generalization of this fact to any bounded strictly convex domain was given by Beardon in [Be97] by an elegant argument using the Hilbert metric. He proved that if $D$ is such a domain then for each positive $\delta$ there is a positive number $M = M(D, \delta)$ such that for any intersecting chords $c_1, c_2$, each of length at least $\delta$, one has

$$M^{-1} \leq \frac{l_1 l_1'}{l_2 l_2'} \leq M\,, \tag{1.1}$$

where $l_1, l_1'$, and $l_2, l_2'$ denote the respective lengths of the segments into which the two chords are divided.