3.2 The proof of Theorem 3.1

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In view of the continuity of κ, the following immediate consequence of Proposition 3.4 is somewhat analogous to a mean value theorem.

COROLLARY 3.5. *Denote by κ_{min} and κ_{max} the minimum and the maximum, respectively, of the curvature of ∂D. Then*

$$\kappa_{min} \leq K(x,y,z) \leq \kappa_{max} \,,$$

for any three boundary points x, y, z.

3.2 THE PROOF OF THEOREM 3.1

Assume that D is as in the theorem. To simplify the notation we will only discuss the 3-dimensional case. Each 2-dimensional plane section is Gromov hyperbolic by the above so we only need an overall bound for constants $\delta(S)$ when S runs through all the plane sections. The intersection of ∂D with a 2-dimensional plane gives rise to a smooth planar curve α, which we assume is parameterized by arclength. The constant δ of the hyperbolicity depends on the curvature of α. These curves could have an arbitrarily large curvature but we need only to bound from above (and hence from below) the ratio of the curvatures at different points of the curve. The curvature vector $\alpha''(t)$ of α at a point $x = \alpha(t)$ lies in this plane and is orthogonal to $\alpha'(t)$. Thus we need to bound the ratio $\frac{|\alpha''(t)|}{|\alpha''(s)|}$. It is a fact (Meusnier's lemma, see [Kl78, p. 43] that

$$k_x(\alpha'(t)) = |\alpha''(t)| \cos\theta(t) \,,$$

where $k_x(\alpha'(t)) = II_x(\alpha'(t), \alpha'(t))$ is the normal curvature in the direction $\alpha'(t)$ and $\theta(t)$ is the angle between $\alpha''(t)$ and the normal of ∂D at x. In view of the assumption (3.1) and Corollary 3.5 we therefore need to bound the ratio $\frac{\cos\theta(s)}{\cos\theta(t)}$ independently of s, t and α. Near any point x the surface ∂D is the graph of a C^2 function $z = f(x,y)$ in suitable Cartesian coordinates. Hence any small plane section C_ε is given by the equation $f(x,y) = \varepsilon > 0$. Expressing θ in terms of f we arrive at the problem of bounding the ratio of the gradients $\frac{|\nabla f(p)|}{|\nabla f(q)|}$ along the section. By rotation in the xy-plane we may assume that the x- and y-axis are along the direction of principal curvature. By developing $f(x,y)$ into a Taylor's expansion around the origin, we obtain $f(x,y) = \frac{1}{2}(ax^2 + by^2) + r$, where r vanishes at $(0,0)$ together with all its derivatives up to second order, and where $a = f_{xx}(0,0), b = f_{yy}(0,0)$ are the principal curvatures. We conclude that $c < \frac{|\nabla f(x,y)|}{\sqrt{x^2+y^2}} < C$ near 0 for universal $c, C > 0$ and thus it remains to bound the ratio $\frac{x^2+y^2}{x'^2+y'^2}$ on C_ε. But this ratio is bounded in view of the estimate $\kappa^{-1}(x^2+y^2) < f(x,y) < \kappa(x^2+y^2)$ for some universal $\kappa > 0$ and of the fact that $f(x,y) = \varepsilon$ on C_ε. □

4. Consequences of Gromov hyperbolicity for the shape of the boundary

PROPOSITION 4.1. *Let D be a bounded convex domain in $\mathbf{R}^n$ and let h be a Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is strictly convex, that is, it does not contain a line segment.*

This can be proven following the proof of N. Ivanov [Iv97] of Masur-Wolf's theorem [MW95] that the Teichmüller spaces (genus ≥ 2) are not Gromov hyperbolic. The proof makes use of Gromov's exponential divergence criterion, see [BH99, p. 412]. For another proof of the above proposition, see [SM00].

THEOREM 4.2. *Let D be a bounded convex domain in $\mathbf{R}^n$ and let h be the Hilbert metric on D. If h is Gromov hyperbolic then the boundary ∂D is smooth of class C^1.*

Proof. *2-dimensional case:* First, by the previous result, D is strictly convex. Let $y = f(x)$, $x \in (-a,a)$ be an equation of ∂D near some point. Then f is strictly convex and hence the one-sided derivatives $f'_-(x)$, $f'_+(x)$ exist and are strictly increasing on $(\varepsilon,\varepsilon)$, [RV73, § 11].

We prove that $f'_-(0) = f'_+(0)$. Suppose not, then by choosing appropriate Cartesian coordinates we may assume that $f'_-(0) < 0$ and $f'_+(0) > 0$. For each sufficiently small ε construct an ideal triangle $\Delta = \Delta(\varepsilon)$ in D with one vertex 0 and two other vertices corresponding to the intersection of the line $y = \varepsilon$ with ∂D. We assert that the slimness of $\Delta(\varepsilon)$ tends to ∞ when ε tends to zero. Namely we show that the Hilbert distance between the point $P = (0,\varepsilon)$ and any point Q of the side $[0,B]$ tends to ∞. Let $f'_+(0) = \tan\alpha$, $0 < \alpha < \pi/2$. Let $x_1 < x_2$ be the points such that $f(x_1) = \varepsilon$ and $f'_+(0)x_2 = \varepsilon$. Then

$$PQ \geq \varepsilon \cos\alpha = f(x_1)\cos\alpha .$$

Let O, R be the intersection points of the line PQ with ∂D. We have therefore

$$QR \leq x_2 - x_1 = \frac{f(x_1)}{f'_+(0)} - x_1 = \frac{f(x_1) - f'_+(0)x_1}{f'_+(0)}$$

and hence, combining the last two inequalities,